Greedy Algorithms

Check in

- Assignment 1 feedback: 100% response rate, which is great!
    - **43% :** just the right amount of challenging, hits a good balance
    - **21% :** too challenging and not in a good way
    - **34% :** somewhere between the above two
    - **0% :** on the easy side, but this is just the beginning
- **Take aways.** Need to modulate…but still challenge (point of class)
    - More practice with proof writing (ongoing process…)
    - More scaffolding around challenging questions
- **Related.** Class make up this semester very different from last
- **Related.** Doing readings from textbook as preparation.

Do You Have Any Questions?

Lecture Proofs vs Written Proofs

- When I present proofs in lecture, my goal is to have in a form that is best for presentation (paired with me being there)
- By design, it is not reflective of how proofs should be written (in a book or a homework assignment)
- Proofs in lectures, will usually be **interactive** and **focused more on providing intuition** than serving as a model for written proofs
- Where should look for a model of written proofs?
    - Textbook! Best resource
    - Homework sample solutions
- Proof writing is a skill, like programming, which you will develop with practice as we go through the semester
- This class is as much about formalization as about algorithms

Greedy: Examples

- We already saw a greedy algorithm that works!
    - Which one?
- Cashier's algorithm to return change in coins?
    - Greedy! To make change for $\$r$, start with biggest denomination less than r, and so on
    - Optimal for US coins!
    - (Not in general)

Greedy: Locally Optimal

Greedy algorithms build solutions by making locally optimal choices

Surprisingly, sometimes this also leads to globally optimal solutions!

We start with greedy algorithms greedy algorithms as the first design paradigm because

- They are natural and intuitive
- Proving they are optimal is the hard part

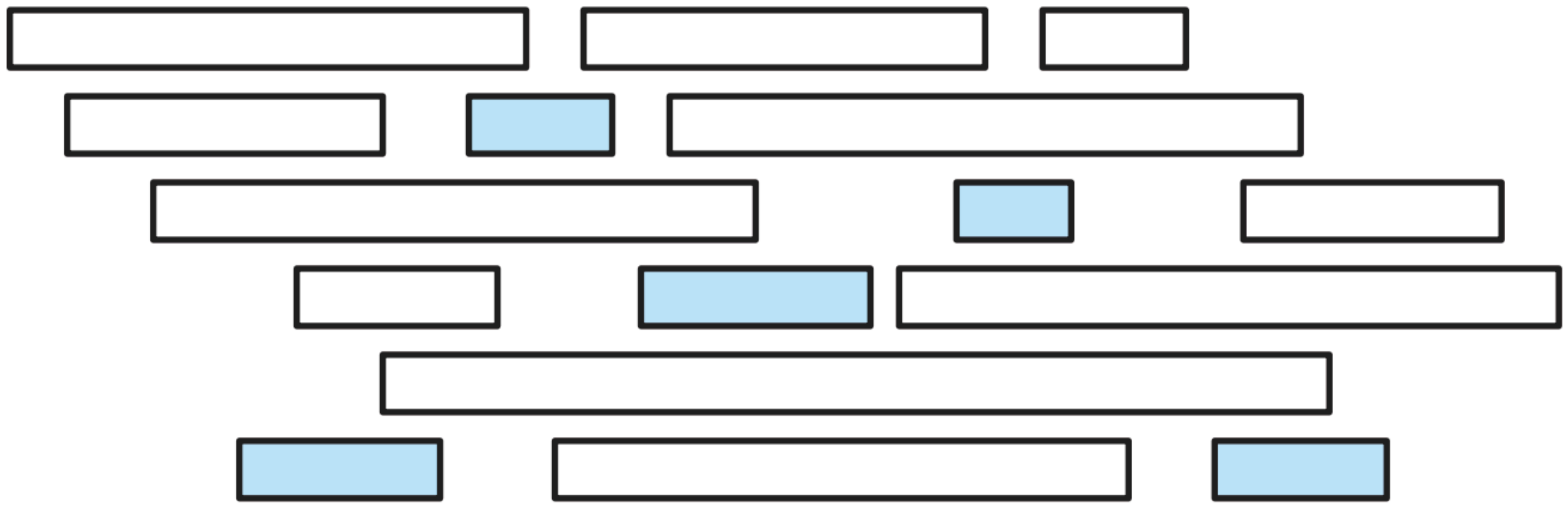
Greedy: Proof Techniques

Two fundamental approaches to proving correctness of greedy algorithms

- **Greedy stays ahead**: Partial greedy solution is, at all times, as good as an "equivalent" portion of any other solution
- **Exchange Property**: An optimal solution can be transformed into a greedy solution without sacrificing optimality.

Class Scheduling

Problem. Given the list of start times $s_1, \ldots, s_n$ and finish times $f_1, \ldots, f_n$ of n classes (labeled $1, \ldots, n$), what is the maximum number of non-conflicting classes you can schedule?

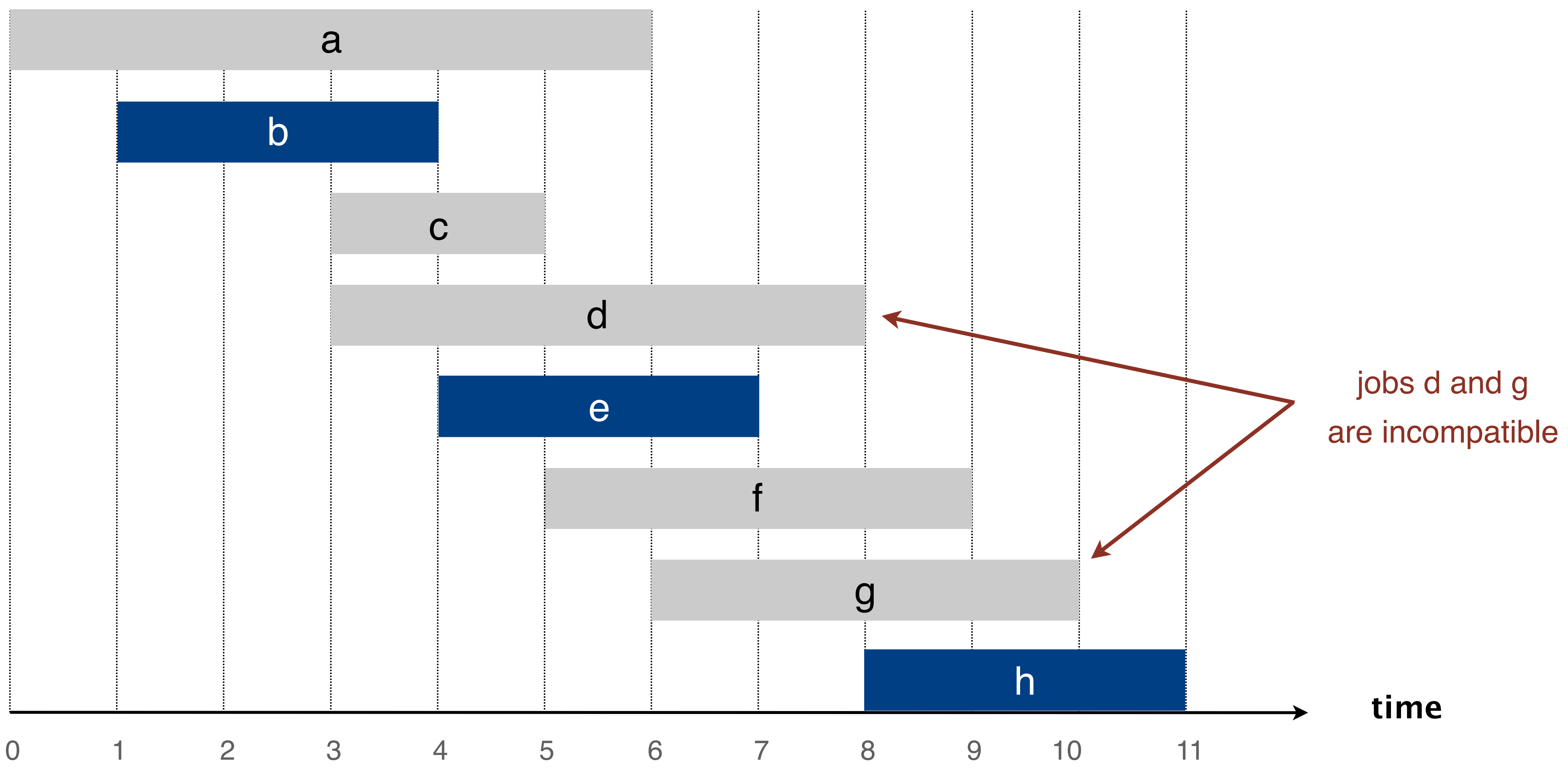


A maximum conflict-free schedule for a set of classes.

From Erickson's Algorithms Book

Interval Scheduling

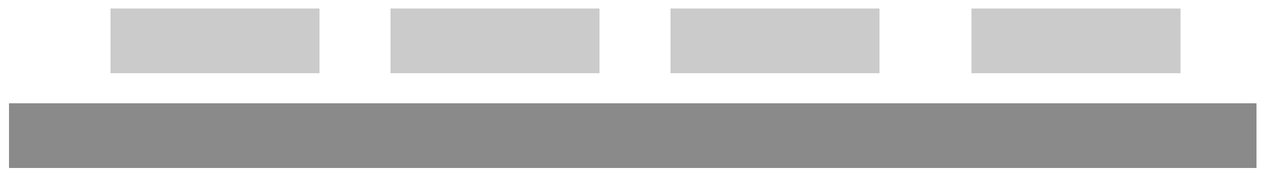
Job scheduling. This is a general job scheduling problem. Suppose you have a machine that can run one job at a time and n job requests with start and finish times: $s_1, \ldots, s_n$ and $f_1, \ldots, f_n$. How to determine the most number of compatible requests?



What to be Greedy About?

- Algorithmic idea: Pick a criterion to be greedy about. Keep choosing compatible jobs based on it
- Lets start with obvious one: **start times**
    - Schedule jobs with **earliest start time** first
- Is this the best way?
    - If not, can we come up with a counter example?

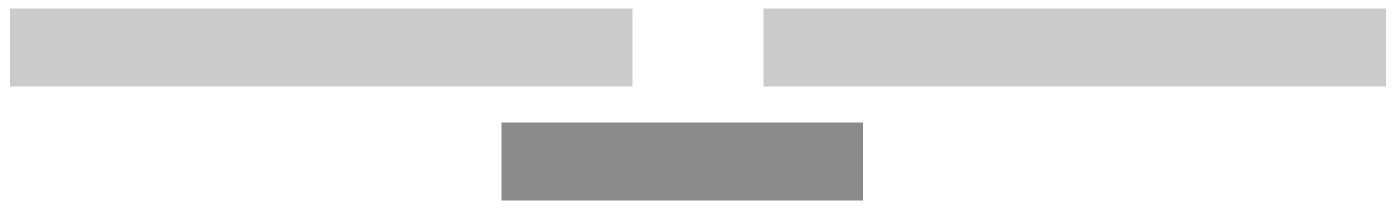
counterexample for earliest start time



Many Ways to be Greedy

- Algorithmic idea: Pick a criterion to be greedy about. Keep choosing compatible jobs based on it
- Another possible criterion:
  - Schedule jobs with **shortest interval** first
  - That is, smallest value of $f_i - s_i$

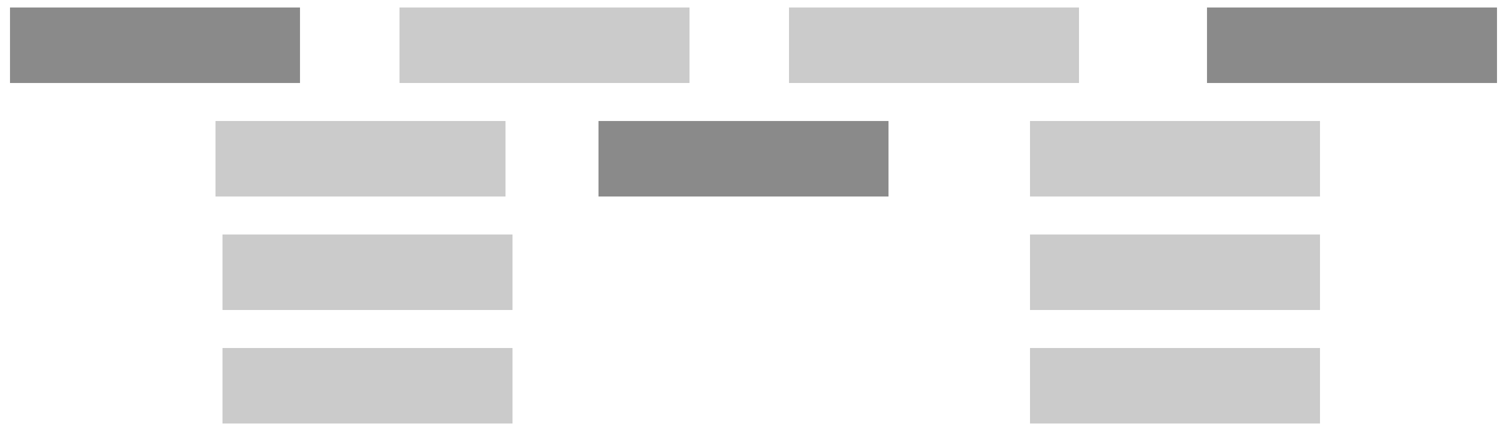
counterexample for shortest interval



Many Ways to be Greedy

- Algorithmic idea: Pick a criterion to be greedy about. Keep choosing compatible jobs based on it
- Another possible criterion:
  - **Fewest conflict**
  - Schedule that conflict with fewest other jobs first

counterexample for fewest conflicts



Many Ways to be Greedy

- Algorithmic idea: Pick a criterion to be greedy about. Keep choosing compatible jobs based on it
- Criteria that do not work:
    - Earliest start time first
    - Shortest interval works
    - Fewest conflict first
- How about: **earliest finish time first?**
    - Surprisingly optimal
    - Need to prove why it is optimal
    - Idea: Free your resource as soon as possible!

Earliest-Finish-Time-First Algorithm

```
EARLIEST-FINISH-TIME-FIRST ($n, s_1, s_2, \ldots, s_n, f_1, f_2, \ldots, f_n$)

SORT jobs by finish times and renumber so that $f_1 \leq f_2 \leq \ldots \leq f_n$.
$S \leftarrow \varnothing$.  ⟵ set of jobs selected
FOR $j = 1$ TO $n$
    IF job $j$ is compatible with $S$
        $S \leftarrow S \cup \{ j \}$.
RETURN $S$.
```

Correctness of Algorithm

- Set S output consists of compatible requests
    - By construction!
- We want to prove our solution S is optimal (schedules the maximum number of jobs)
- Let $\mathcal{O}$ be an optimal set of jobs. **Goal:** show $|S| = |\mathcal{O}|$,i.e., greedy also selects the same number of jobs and thus is optimal
- **Proof technique to prove optimality:**
    - Greedy always "stays ahead" (or rather never falls behind)
    - We will compare partial solutions of greedy vs an optimal and show that greedy is doing better or just as well
    - **Intuition:** greedy frees up the resource as soon as possible
    - Lets use this metric to compare greedy and optimal

Get Ahead Stay Ahead Proof

Correctness proof. Let $g_1, \ldots, g_k$ and $o_1, \ldots, o_m$ be the sequence of compatible jobs selected by the greedy and optimal algorithm respectively, ordered by increasing finish time.

Lemma 1. For all $i \leq k$, we have: $f_{g_i} \leq f_{o_i}$.

Proof. (By induction) Base case: $i = 1$ is true, why?

- Assume holds for $k-1$: $f_{g_{k-1}} \leq f_{o_{k-1}}$
- For kth job, note that $f_{o_{k-1}} \leq s_{o_k}$ (why?)
- Using inductive hypothesis: $f_{g_{k-1}} \leq f_{o_{k-1}} \leq s_{o_k}$
- Greedy picks earliest finish time among compatible jobs (which includes o_k) thus $f_{g_k} \leq f_{o_k}$

∎

Are We Done? Almost

Let $g_1, \ldots, g_k$ and $o_1, \ldots, o_m$ be the sequence of compatible jobs selected by the greedy and optimal algorithm respectively, ordered by finish times.

Lemma 1. For all $i \leq k$, we have: $f_{g_i} \leq f_{o_i}$.

Lemma 2. The greedy algorithm returns an optimal set of jobs S, that is, $k = m$.

Proof. (By contradiction)

Suppose S is not optimal, then the optimal set $\mathcal{O}$ must select more jobs, that is, $m > k$.

That is, there is a job o_{k+1} that starts after o_k ends

What is the contradiction? Greedy keeps selecting jobs until no more compatible jobs left. Since $f_{g_k} \leq f_{o_k}$ by Lemma 1, greedy would also select compatible job o_{k+1} ($\Rightarrow\Leftarrow$) ∎

Implementation & Running Time

Analysis (Running time):

Let’s analyze all the steps:

- Sorting jobs by finish times
    - $O(n \log n)$
- Permuting start times in the order of finish times
    - $O(n)$
- For each selected job i, find next job j such that $s_j \geq f_i$
    - Iterate through the list until you reach the right interval j
    - This part of the algorithm is $O(1)$ per interval, so $O(n)$
- Overall $O(n \log n)$ time

Greedy Algorithms: Class Quiz

Question.

- Suppose that each job also has a positive weight and the goal is to find a maximum weight subset of mutually compatible intervals.
- Is the earliest-finish-time-first algorithm still optimal?
- If no, can we design a simple counter example?

Minimizing Lateness: Problem

Given: A list of processes needs to be scheduled

- Only one process can be executed at a time
- A process must run to completion before another can be executed
- Each process has a duration t_i and a deadline d_i

Goal: Schedule tasks: $(t_i, d_i) \rightarrow (s_i, f_i)$ (start & finish times), where $f_i = s_i + t_i$, to minimize maximum lateness

- Satisfy all requests but optimize max lateness
- Lateness of process i : $L_i = \max\{0, f_i - d_i\}$
- Resource is first available at time 0

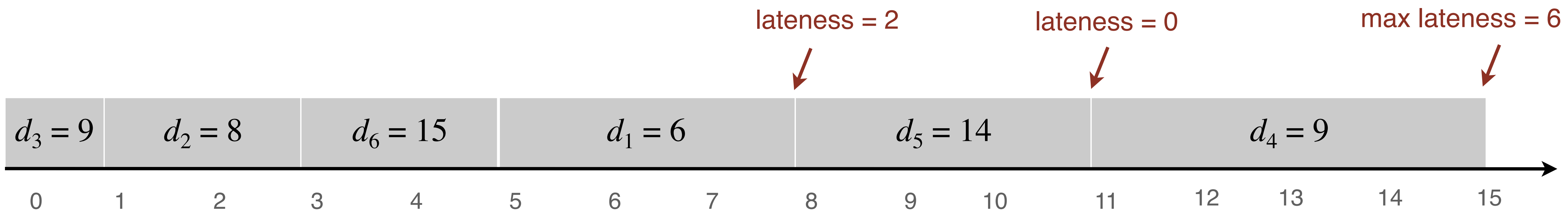
Minimizing Lateness: Problem

Given: A list of processes needs to be scheduled, each process has a duration t_i and a deadline d_i

Goal: Schedule tasks: $(t_i, d_i) \to (s_i, f_i)$ (start & finish times), where $f_i = s_i + t_i$, to minimize maximum lateness

- Lateness of process i : $L_i = \max\{0, f_i - d_i\}$

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



Minimizing Lateness: Problem

Possible strategies?

- Shortest jobs first (get more done faster!)
- Do jobs with shortest slack time first (slack of job i is $d_i - t_i$)
- Earlier deadlines first (triage!)

Shortest job first:

	1	2
t_j	1	10
d_j	100	10

counterexample

Gives max lateness: 1
OPT max lateness: 0

Minimizing Lateness: Problem

Possible strategies

- Shortest jobs first (get more done faster!)
- Do jobs with shortest slack time first (slack of job i is $d_i - t_i$)
- Earlier deadlines first (triage!)

Shortest slack first:

	1	2
t_j	1	10
d_j	2	10

counterexample

Gives max lateness: 9
OPT max lateness: 1

Minimizing Lateness: Problem

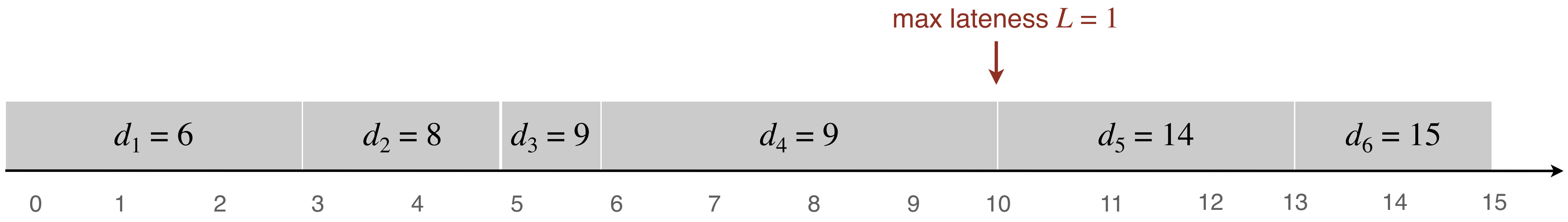
Possible strategies

- Shortest jobs first (get more done faster!)
- Do jobs with shortest slack time first (slack of job i is $d_i - t_i$)
- Earlier deadlines first (triage!)

- How all computer scientists schedule their work
- Order jobs by their deadline and schedule them in that order
- **Intuition:** get the jobs due first done first
- Surprisingly optimal (We will show this)
- Disregards job lengths! (Seems counter-intuitive)

Earliest Deadline First

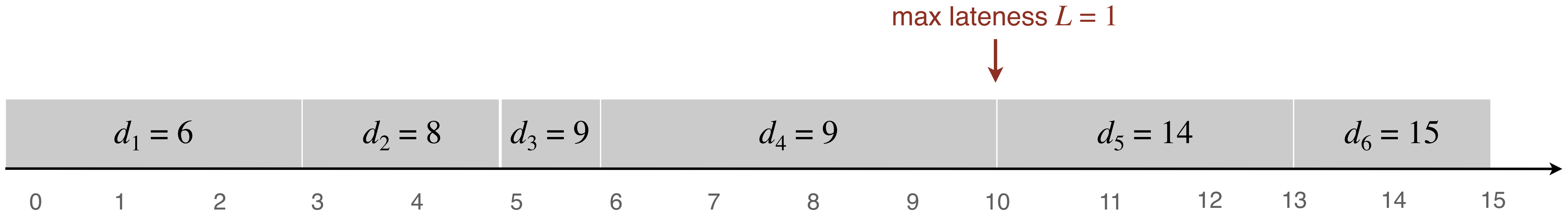
```
EARLIEST-DEADLINE-FIRST ($n, t_1, t_2, \ldots, t_n, d_1, d_2, \ldots, d_n$)

SORT jobs by due times and renumber so that $d_1 \le d_2 \le \ldots \le d_n$.
$t \leftarrow 0$.
FOR $j = 1$ TO $n$
    Assign job $j$ to interval $[t, t + t_j]$.
    $s_j \leftarrow t$ ;  $f_j \leftarrow t + t_j$.
    $t \leftarrow t + t_j$.
RETURN intervals $[s_1, f_1], [s_2, f_2], \ldots, [s_n, f_n]$.
```



Recall: Scheduling with Deadlines

Given interval length t_i and deadline d_i for $i \in \{1, \ldots, n\}$ jobs, schedule all tasks, that is, assign start and finish times $(t_i, d_i) \to (s_i, f_i)$, where $f_i = s_i + t_i$, so as to minimize the maximum lateness.

- Lateness of process i : $L_i = \max\{0, f_i - d_i\}$



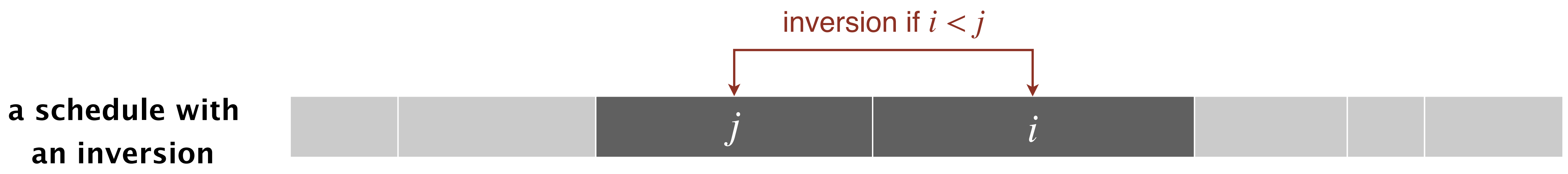
Greedy: Earliest-Deadline First

```
EARLIEST-DEADLINE-FIRST $(n, t_1, t_2, \ldots, t_n, d_1, d_2, \ldots, d_n)$

SORT jobs by due times and renumber so that $d_1 \le d_2 \le \ldots \le d_n$.
$t \leftarrow 0$.
FOR $j = 1$ TO $n$
    Assign job $j$ to interval $[t, t + t_j]$.
    $s_j \leftarrow t$ ; $f_j \leftarrow t + t_j$.
    $t \leftarrow t + t_j$.
RETURN intervals $[s_1, f_1], [s_2, f_2], \ldots, [s_n, f_n]$.
```

Minimizing Lateness: Greedy

Observations about our greedy algorithm

- It produces a schedule with no idle time
- It produces a schedule with no inversions
    - i, j is an inversion if job j is scheduled before i but i's deadline is earlier ($d_i < d_j$)



recall: we assume the jobs are numbered so that $d_1 \leq d_2 \leq \ldots \leq d_n$

Structure of the Solution

- **Notice:** All schedules with no inversions and no idle time have the same maximum lateness
  - Distinct deadlines, unique schedule
  - Non-distinct deadlines: Consider two jobs with deadline d; the maximum lateness does not depend on the order in which they are scheduled
  - Say the two jobs have duration t_i, t_j and same deadline d
  - If i is scheduled first at time s, the max lateness is: $\max\{0, (s + t_i + t_j) - d\}$
  - If j is scheduled first at time s, the max lateness is the same: $\max\{0, (s + t_i + t_j) - d\}$

Continued in Next Lecture

Acknowledgments

- The pictures in these slides are taken from
  - Kleinberg Tardos Slides by Kevin Wayne (https://www.cs.princeton.edu/~wayne/kleinberg-tardos/pdf/04GreedyAlgorithmsI.pdf)
  - Jeff Erickson's Algorithms Book (http://jeffe.cs.illinois.edu/teaching/algorithms/book/Algorithms-JeffE.pdf)