Programming Language Notes

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1 Introduction

A well-written program is a poem. Both are powerful because their content is condensed without being inscruitable and because the form is careful chosen to give insight into the topic. For a program, the topic is an algorithm and the implementation should emphasize the key steps while minimizing the details. The most elegant implementation is not always the most efficient, although it often is within a constant factor of optimal. The choice of programming language most closely corresponds to the choice of poem *structure*, e.g., sonnet or villanelle, not the choice of natural language, e.g., English or French. Structure enforces certain patterns and ways of thinking on the author and reader, thus aiding certain kinds of expression and inhibiting others.

To author elegant programs, one must master a set of languages and language features. Then, one must subjectively but precisely choose among them to express specific algorithms. Languages are themselves designed. A well-designed *language* is a meta-poem. A language designer crafts a set of expressive tools suited to the safety, performance, and expressive demands of a problem domain. As with literature, the difficult creative choice is often not what to include, but what to omit.

A programming language is a mathematical calculus, or **formal language**. Its goal is to express algorithms in a manner that is unambiguous to people and machines. Like any calculus, a language defines both syntax and semantics. Syntax is the grammar of the language; the notation. Semantics is the meaning of that notation. Since syntax can easily be translated, the semantics are more fundamental.

Church and Turing (and Kleene) showed that the minimal semantics of the λ calculus and Turing machine are sufficient to emulate the semantics of any more complicated programming language or machine. However, reducing a particular language to the λ calculus may require holistic restructuring of programs in that language.

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We say that a particular language feature (e.g., continuations, macros, garbage collection) is **expressive** if it cannot be emulated without restructuring programs that use it.

In these notes, **features** are our aperture on programming languages. These features can increase and decrease the expressiveness of the language for certain domains. Why would we want to decrease expressiveness? The primary reason is to make it easier to automatically reject invalid programs, thus aiding the debugging and verification process. Compilers print error messages when they automatically detected (potential) errors in your program. You might have previously considered such errors to be bad. But they are good! When this happens, the compiler saves you trouble of finding the errors by testing (as well as the risk of *not* finding them at all). A major challenge and theme of programming language design is simultaneously pushing the boundaries of what is checkable and expressive in a language.

1.1 Types

Every language makes some programs easy to express and others difficult. When a language is well-suited to a problem domain, the programs it makes easy to express are correct solutions to problems in that domain. A well-suited language furthermore makes it hard to express programs that are incorrect. This is desirable! One way to design a language is to selectively add restrictions until it is hard to express incorrect programs for the target domain. The cost of a language design is that some correct and potentially useful programs also become hard to express in the language.

The **type system** is one tool for restricting a language. A type system associates metadata with values and the variables that can be bound to them. A well-typed program is one where constraints on the metadata imposed by the language and program itself are satisfied. When these are violated, e.g., by assigning a "String" value to an "int" variable in Java, the program is incorrect. Some kinds of program errors can be detected by static analysis, which means examining the program without executing it. Some kinds of errors cannot be detected efficiently through static analysis, or are statically undecidable. Many of these can be detected by dynamic analysis, which means executing type checks at **run-time**–while the program is executing.

We say that a language exhibits **type soundness** if well-typed programs in that language cannot "go wrong" [Mil78]. That is, if well-typed programs cannot reach stuck states [WF94] from which further execution rules are undefined. Another view of this is that "A language is **type-safe** if the only operations that can be performed on data in the language are those sanctioned by the type of the data." [Sar97]

By declaring undesirable behaviors–such as dereferencing a null pointer, accessing a private member of another class, or reading from the filesystem–to be type errors and thus unsanctioned, the language designer can leverage type soundness to enforce safety and security.

All languages (even assembly languages) assign a type to a value at least before it is operated on, since operations are not well-defined without an interpretation of the data. Most languages also assign types to values that are simply stored in memory. One set of languages that does not is assembly languages: values in memory (including registers) are just bytes and the programmer must keep track of their interpretation implicitly. **Statically typed** languages contain explicit declarations that limit the types of values a to which a variable may be bound. C++ and Java are statically typed languages. **Dynamically typed** languages such as Scheme and Python allow a variable to be bound to any type of value. Some languages, like ML, are dynamically typed but the interpreter uses **type inference** to autonomously assign static types where possible.

1.2 Imperative and Functional

The discipline of computer science grew out of mathematics largely due to the work of Church and his students, particularly Turing. Church and Kleene created a mathematical system called the λ **calculus** (also written out as the lambda calculus) that treats mathematical functions as first-class values within mathematics. It is minimalist in the sense that it contains the fewest possible number of expressions, yet can encode any decidable function. Turing created the **Turing machine** abstraction of a minimal machine for performing computations. These were then shown to be equivalent an minimal models of computation, which is today called the **Church-Turing Thesis**.

These different models of computation are inherited by different styles of programming. Turing's machine model leads to **imperative programming**, which operates by mutating (changing) state and proceeds by iteration. Java and C++ are languages that encourage this style. Church's mathematical model leads to **functional programming**, which operates by invoking functions and proceeds by recursion. Scheme, ML, Unix shell commands, and Haskell are languages that encourage this style. So-called scripting languages like Python and Perl encourage blending of the two styles, since they favor terseness in all expressions.

2 Computability

2.1 The Incompleteness Theorem

At the beginning of the 20th century, mathematicians widely believed that all true theorems could be reduced to a small set of axioms. The assumption was that mathematics was sufficiently powerful to prove all true theorems. Hilbert's program[1] was to actually reduce the different fields of mathematics to a small and consistent set of axioms, thus putting them all on a solid and universal foundation.

In 1931 Gödel [G3̈1][vH67, 595] proved that in any sufficiently complex system of mathematics (i.e., formal language capable of expressing at least arithmetic), there exist true statements that cannot be proven using that system, and that the system is therefore incomplete (unable to prove its own consistency). This **Incompleteness Theorem** was a surprising result and indicated that a consistent set of axioms could not exist. That result defeated Hilbert's program[2] and indicated for the first time the limitations of mathematics. This is also known as the First Incompleteness Theorem; there is a second theorem that addresses the inconsistency of languages that claim to prove their own consistency.

Here is a proof of the Incompleteness Theorem following Gödel's argument. Let every statement in the language be encoded by a natural number, which is the **Gödel Number** of that statement. This encoding can be satisfied by assigning every operator, variable, and constant to a number with a unique prefix and then letting each statement be the concatenation of the digits of the numbers in it. (This is roughly equivalent to treating the text of a program as a giant number containing the concatenation of all of its bits in an ASCII representation.) For example, the statement "$x > 4$" might be encoded by number g:

$$S_g(x) = \text{“}x > 4\text{”} \tag{1}$$

Now consider the self-referential ("recursive") statement,

$$S_i(n) = \text{“}S_n \text{ is not provable.”} \tag{2}$$

evaluated at $n = i$. This statement is a formal equivalent of the **Liar's Paradox**, which in natural language is the statement, "This sentence is not true." $S_n(n)$ creates an inconsistency. As a paradox, it can neither be proved (true), nor disproved (false).

As a result of the Incompleteness Theorem, we know that there exist functions whose results cannot be computed. These **non-computable functions** (also called **undecidable**) are interesting for computer science because they indicate that there are mathematical statements whose validity cannot be determined mechanically. For computer science, we define computability as:

> A function f is **computable** if there exists a program P that computes f, i.e., for any input x, the computation $P(x)$ halts with output $f(x)$.

Unfortunately, many of undecidable statements are properties of programs that we would like a compiler to check. A constant challenge in programming language development is that it is mathematically impossible to prove certain properties about arbitrary programs, such as whether a program does not contain an infinite loop.

2.2 The Halting Problem

Let the **Halting Function** $H(P,x)$ be the function that, given a program P and an input x to P, has value "halts" if $P(x)$ would halt (terminate in finite time) were it to be run, and has value "does not halt" otherwise (i.e., if $P(x)$ would run infinitely, if run). The Halting Problem is that of solving H; Turing [Tur36] proved in 1936 that H is undecidable in general.

[1] "program" as in plan of action, not code

[2] ...and answered Hilbert's "second problem": prove that arithmetic is self-consistent. Whitehead and Russell's *Principia Mathematica* previously attempted to derive all mathematics from a set of axioms.

Theorem 1. *$H(P,x)$ is undecidable.*

Proof. Assume program $Q(P,x)$ computes H (somehow). Construct another program $D(P)$ such that

```
$D(P)$:
    if $Q(P,P)$ = "halts" then loop
    else halt
```

In other words, $D(P)$ exhibits the opposite halting behavior of $P(P)$.

Now, consider the effect of executing $D(D)$. According to the program definition, $D(D)$ must halt if $D(D)$ would run forever, *and* $D(D)$ must run forever if $D(D)$ would halt. Because $D(D)$ cannot both halt and run forever, this is a contradiction. Therefore the assumption that Q computes H is false. We made no further assumption beyond H being decidable, therefore H must be undecidable. □

The proof only holds when H must determine the status of every program and every input. It *is* possible to prove that a specific program with a specific input halts. For a sufficiently limited language, it is possible to solve the Halting Problem. For example, every finite program in a language without recursion or iteration must halt.

The theorem and proof can be extended to most observable properties of programs. For example, within the same structure one can prove that it is undecidable whether a program prints output or reaches a specific line in execution. Note that it is critical to the proof that $Q(P,x)$ does not actually run P; instead, it must decide what behavior P *would* exhibit, were it to be run, presumably by examining the source code of P. See http://www.cgl.uwaterloo.ca/ csk/halt/ for a nice explanation of the Halting Problem using the C programming language. Two straightforward ways to prove that a property is undecidable are:

- Show that the Halting Problem reduces to this property. That is, if you can solve static checking of the property, then you can solve the Halting Problem, therefore the property is at least as hard as the Halting Problem and is undecidable.
- Rewrite a Halting Problem proof, substituting the property for halting. Note that this is not the same as reducing the property to the Halting Problem.

2.3 Significance of Decidability

Language designers are faced with a dilemma because, like halting, most properties of a program in a sufficiently powerful language are undecidable. One choice is to abandon checking certain properties until run time. We call run-time checking **dynamic checking**. The enables a language to easily express many kinds of programs, however, it means that the programmer has little assurance that the program is actually correct. Only extensive testing will find errors in such programs, and extensive testing is expensive, time consuming, and impractical for programs with large branch factors. Python and Scheme are two languages that defer almost all checking until run time. It is easy to write programs in these languages and hard to find errors in them.

Another choice is to restrict the expressive power of the language somewhat, so that more properties can be checked before a program is run. This is called **static checking** (a.k.a. **compile-time checking**). This makes it harder to construct programs, however it enables much stronger guarantees than testing can provide– the checker can *prove* that a program does not have certain kinds of errors, without ever running it. C++ and ML are languages that provide significant static checking facilities. A language like Java is somewhere in between. It performs some checks statically, but in order to make the language more flexible the designers made many other checks dynamic. The one of the most common check is the null-pointer dereference, which many programmers will recognize as their most common error as well. Java programmers may be pleasantly surprised to discover that there are many languages (unfortunately none of the popular ones mentioned in this paragraph) in which most null-pointer checks can be made statically, and therefore appear very infrequently as run-time errors.

A third choice is to restrict the expressive power of the language so severely that certain kinds of errors are simply not expressible. This is useful for metalanguages, like type systems, and for embedded languages, like early versions of the 3D graphics DirectX HLSL and OpenGL GLSL languages. Once can guarantee that programs in such languages always halt[3], for example, by simply not providing the mechanisms for function calls or variable-length loops. Of course, programmers often find those restrictions burdensome and most practical languages eventually evolve features that expand the expressive power to defeat their static checkability. This has already occurred in the C++ type system with the template feature and in both the aforementioned HLSL and GLSL languages.

3 Life of a Program

A program goes through three major stages: Source, Expressions, and Values. Formal specifications describe the syntax of the source and the set of expressions using an **grammar**, typically in BNF. This is called the **expression domain** of the language. The **value domain** is described in set notation or as BNF grammars. Expressions are also called **terms**. Expressions that do not reduce to a value are sometimes called **statements**.

An analogy to a person reading a book helps to make clear the three stages. The physical ink on the printed page is source. The reader scans the page, distinguishing tokens of individual letters and symbols from clumps of ink. In their mind, these are assigned the semantics of words–i.e., expressions. When those expressions are evaluated, the value (meaning) of the words arises in the readers mind. This distinction is subtle in the case of literals. Consider a number written on the page, such as “32”. The curvy pattern of ink is the source. The set of two digits next to each other is the expression. The interpretation of those digits in the reader’s mind is the number value. The number value is not something that can be written, because the act of writing it down converts it back into an expression. Plato might say that the literal expression is a shadow on the cave wall of the true value, which we can understand but not directly observe. [4]

3.1 Source Code and Tokens

A program begins as **source code**. This is the ASCII (or, increasingly, unicode!) string describing the program, which is usually in a file stored on disk. A **tokenizer** converts the source to a stream of **tokens** in a manner that is specific to the language. For example, in Java the period character “.” becomes a separate token if it separates two identifiers (variables) but is part of a floating-point number if it appears in the middle of a sequence of digits, e.g., string.length() versus 3.1415. See java.StringTokenizer or G3D::TextInput for an example of an implementation.

Figures 3.1 and 3.1 show an example of the source code and resulting token stream for a simple factorial function implemented in the Scheme programming language. The tokenizer is often language-specific. For this example, the tokenizer tags each token as a parenthesis, reserved word, identifier, or numeral. Source code is usually stored in a string. A typical data structure for storing the token stream is an array of instances of a token class.

```
(define (factorial n)
     (if (< n 2)
           ; Base:
          1
           ; Recurse:
          (* n (factorial (- n 1)))))
```

Figure 1: Scheme source code for factorial.

[3]A language in which programs always halt is called **strongly normalizing**.

[4]For the truly philosophical, what is in the mind, or what is stored in bits in a computer’s memory, is still only a *representation* of the value. The actual *number* that the *numeral* 32 represents is unique. There can be only one 32, which means it can’t be in multiple places at once–the bits representing the numeral 32 in a computer’s memory therefore act a pointer to the ideal number 32. AI, PL, and philosophy meet when we consider whether the human mind is different, or just shuffling around around representations like a computer.

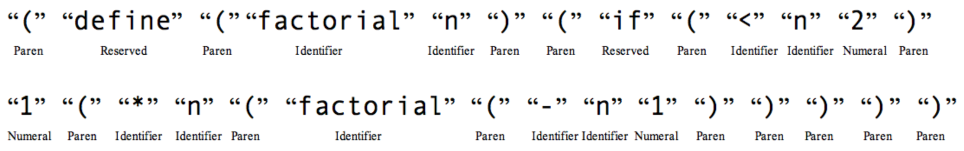

"(" "define" "(" "factorial" "n" ")" "(" "if" "(" "<" "n" "2" ")"
Paren Reserved Paren Identifier Identifier Paren Paren Reserved Paren Identifier Identifier Numeral Paren
"1" "(" "*" "n" "(" "factorial" "(" "-" "n" "1" ")" ")" ")" ")" ")"
Numeral Paren Identifier Identifier Paren Identifier Paren Identifier Identifier Numeral Paren Paren Paren Paren Paren

Figure 2: Token stream for factorial.

3.2 Expressions

A **parser** converts the token stream into a **parse tree** of **expressions**. The legal expressions are described by the **expression domain** of the language, which is often specified in **BNF**. The nodes of a parse tree are instances of expressions (e.g., a `FOR` node, a `CLASS-DEFINITION` node) and their children are the sub-expressions. The structure of the parse tree visually resembles the indenting in the source code. Figure 3.2 shows a parse tree for the expressions found in the token stream from figure 3.1.

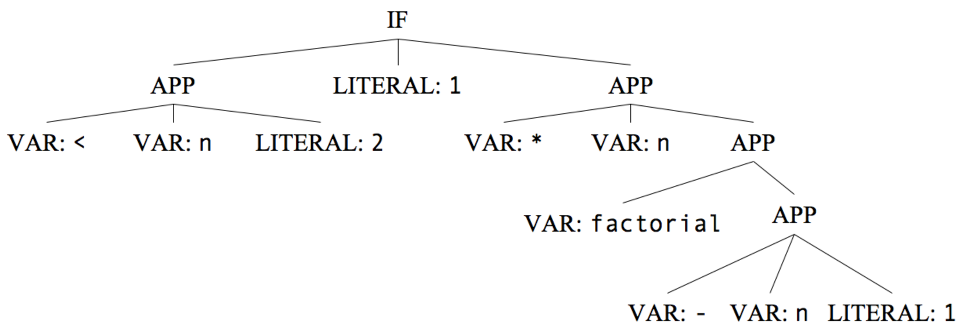


Figure 3: Parse tree for factorial.

The Scheme language contains the `QUOTE` special form for conveniently specifying parse trees directly as literal values, omitting the need for a tokenizer and parser when writing simple interpreters for languages that have an abstract syntax. The drawback of this approach is that simply quoting the factorial code in figure 3.1 would not produce the tree in figure 3.2. Instead, the result would be a tree of symbols and numbers without appropriate expression types labeling the nodes.

3.3 Values

When the program executes (if compiled, or when it is evaluated by an interpreter if not), expressions are reduced to values. The set of legal values that can exist during execution is called the **value domain**. The value domain typically contains all of the first-class values, although some languages have multiple value domains and restrict what can be done to them. In general, a value is **first-class** in a language if all of the following hold:

1. The value can be returned from a function
2. The value can be an argument to a function
3. A variable can be bound to the value
4. The value can be stored in a data structure

Java generics (a polymorphism mechanism) do not support primitive types like `int`, so in some sense those primitives are second-class in Java and should be specified in a separate domain from `Object` and its subclasses, which are first-class. In Scheme and C++, procedures (functions) and methods are first-class because all of the above properties hold. Java methods are not first-class, so that language contains a `Method` class that describes a method and acts as a proxy for it.

The value domain can be specified using set notation, e.g.,

$$\begin{aligned} real &= int \cup decimal \\ complex &= real \times real \\ number &= real \cup complex \end{aligned}$$

or using a BNF grammar (at least, for a substitution interpreter), which is described later.

3.4 Implementation Issues

There is a design tradeoff when implementing a language between compactness and abstraction. Using the same types in the implementation language for source, expressions, and values reduces the amount of packing and unpacking of values that is needed, and allows procedures in the implementation language to operate directly on the values in the target language. Furthermore, in Scheme, the READ procedure and QUOTE special form allow easy creation of tree values using literals that are syntactically identical to Scheme source code. This avoids the need for an explicit tokenizer and parser. Using the same types across domains violates the abstraction of those domains. This can make the implementation of the language harder to understand (when it grows large), and limits the ability of the type checker to detect errors in the implementation. For example, when implementing a Scheme interpreter in Java, one could choose to implement Scheme symbols, strings, identifiers, and source all as Java strings, without a wrapper class to distinguish them. It would be easy to accidentally pass a piece of source code to a method that expected an identifier, and the Java compiler could not detect that error at compile time because the method was only typed to expect a String, not a SchemeIdentifier.

4 Interpreters and Compilers

A compiler is a program that translates other programs in a high-level language to the machine language of a specific computer. The result is sometimes called a **native** binary because it is in the native language of the computer[5]. An interpreter is a program that executes other programs without compiling them to native code. There is a wide range of translation within the classification of interpreters. At one end of this range, some interpreters continuously re-parse and interpret code as they are moving through a program. At the other end, some interpreters essentially translate code down to native machine language at runtime so that the program executes very efficiently.

Although most languages can be either compiled or interpreted, they tend to favor only one execution strategy. C++, C, Pascal, Fortran, Algol, and Ada are typically compiled. Scheme, Python, Perl, ML, Matlab, JavaScript, HTML, and VisualBasic are usually interpreted. Java is an interesting case that compiles to machine language for a computer that does not exist. That language is then interpreted by a virtual machine (JVM).

Compilers tend to take advantage of the fact that they are run once for a specific instance of a program and perform much more static analysis. This allows them to produce code that executes efficiently and to detect many program errors at compile time. Detecting errors before a program actually runs is important because it reduces the space of possible runtime errors, which in turn increases reliability. Compiled languages often have features, such as static types, that have been added specifically to support this kind of compile-time analysis.

Interpreters tend to take advantage of the fact that code can be easily modified while it is executing to allow extensive interaction and debugging of the source program. This also makes it easier to patch a program without halting it, for example, when upgrading a web server. Many interpreted languages were designed with the knowledge that they would not have extensive static analysis and therefore omit the features that would support it. This can increase the likelihood of errors in the programs, but can also make the source code more readable and compact. Combined with the ease of debugging, this makes interpreted languages often feel "friendlier" to the programmer. This typically comes at the cost of decreased runtime performance cost increased runtime errors.

Compiled programs are favored for distributing proprietary algorithms because it is hard to reverse engineer a high-level algorithm from machine language. Interpreted programs by their nature require that the source be

[5]Although in practice, most modern processors actually emulate their published interface using a different set of operations and registers. This allows them include new architectural optimizations without changing the public interface, for compatibility.

distributed, although it is possible to obfuscate or, in some languages, encrypt the source to discourage others from reading it.

5 Syntax

Although we largely focus on semantics, some notable points about syntax:

- A parser converts source code to expressions
- Backus-Naur Form (BNF) formal grammars are a way of describing syntax using recursive patterns
- **Infix** syntax places an operator between its arguments, e.g., "`1 + 2`". Java uses infix syntax for arithmetic and member names, but prefix syntax for method application.
- Prefix syntax places the operator before the operands, e.g., "`add(1, 2)`", which conveniently allows more than one or two operands and unifies operator and function syntax. Scheme uses prefix syntax for all expressions.
- Postfix places the operator after the operands, which allows nested expressions where the operators take a fixed number of arguments, without requiring parentheses. Postscript and some calculators use postfix.
- Scheme's "abstract syntax" makes it easy to parse
- **Macros** allow a programmer to introduce new syntax into a language
- Python has an interesting syntax in which whitespace is significant. This reduces visual clutter but makes the language a little difficult to parse and to edit (in some cases)
- **Syntactic sugar** makes a language sweeter to use without increasing its expressive power

5.1 Backus-Naur Form

Backus-Naur Form (**BNF**) is a formal way of describing context-free grammars for formal languages. A grammar is **context-free** when the the grammar is consistent throughout the entire language (i.e., the rules don't change based on context). BNF was first used to specify the ALGOL programming language.

Beyond its application to programming language syntax, BNF and related notations are useful for representing the grammars of any kind of structured data. Examples include file formats, types, database records, and string search patterns.

A BNF grammar contains a series of rules (also known as **productions**). These are patterns that legal programs in the specified language must follow. The patterns are typically recursive. In the BNF syntax, the **nonterminal** being defined is enclosed in angular brackets, followed by the "::=" operator, followed by an expression pattern. The expression pattern contains other nonterminals, terminals enclosed in quotation marks, and the vertical-bar operator "|" that indicates a choice between two patterns. For example,

$$\begin{aligned} \langle digit \rangle &::= \text{‘0’} \mid \text{‘1’} \mid \text{‘2’} \mid \text{‘3’} \mid \text{‘4’} \mid \text{‘5’} \mid \text{‘6’} \mid \text{‘7’} \mid \text{‘8’} \mid \text{‘9’} \\ \langle digits \rangle &::= \langle digit \rangle \mid \langle digit \rangle \langle digits \rangle \end{aligned}$$

In this document, these are typeset using an unofficial (but common) variation, where terminals are typeset as `x` and nonterminals as *x*. This improves readability for dense expressions. With this convention, digits are:

$$\begin{aligned} digit &::= \texttt{0} \mid \texttt{1} \mid \texttt{2} \mid \texttt{3} \mid \texttt{4} \mid \texttt{5} \mid \texttt{6} \mid \texttt{7} \mid \texttt{8} \mid \texttt{9} \\ digits &::= digit \mid digit\ digits \end{aligned}$$

It is common to extend BNF with regular expression patterns to avoid the need for helper productions. These include the following notation:

(*x*) = *x*; parentheses are for grouping only

[*x*] = zero or one instances of *x* (i.e., *x* is optional)

x^* = zero or more instances of x

x^+ = one or more instances of x

An example of these patterns for expressing a simple programming language literal expression domain (e.g., a subset of Scheme's literals):

$$
\begin{aligned}
\textit{boolean} &::= \texttt{\#t} \mid \texttt{\#f} \\
\textit{digit} &::= \texttt{0} \mid \texttt{1} \mid \texttt{2} \mid \texttt{3} \mid \texttt{4} \mid \texttt{5} \mid \texttt{6} \mid \texttt{7} \mid \texttt{8} \mid \texttt{9} \\
\textit{integer} &::= [\ \texttt{+} \mid \texttt{-}\ ]\ \textit{digit}^+ \\
\textit{rational} &::= \textit{integer}\ \texttt{/}\ \textit{digit}^+ \\
\textit{decimal} &::= [\ \texttt{+} \mid \texttt{-}\ ]\ \textit{digit}^*\ \texttt{.}\ \textit{digit}^+ \\
\textit{real} &::= \textit{integer} \mid \textit{rational} \mid \textit{decimal}
\end{aligned}
$$

BNF can be applied at both the **character level** (e.g., to describe a lexer/tokenizer) and the **token level** (e.g., to describe a parser). The preceding example operates on individual characters within a string and is useful to a tokenizer. An example of a subset of Scheme's expression domain represented in BNF at the token level is:

$$
\begin{aligned}
\textit{variable} &::= \textit{id} \\
\textit{let} &::= \texttt{(}\ \texttt{let}\ \texttt{(}\left(\ \texttt{[}\ \textit{id exp}\ \texttt{]}\ \right)^*\ \texttt{)}\ \textit{exp}\ \texttt{)} \\
\textit{lambda} &::= \texttt{(}\ \texttt{lambda}\ \texttt{(}\ \textit{id}^*\ \texttt{)}\ \textit{exp}\ \texttt{)} \\
\textit{exp} &::= \textit{variable} \mid \textit{let} \mid \textit{lambda}
\end{aligned}
\tag{3}
$$

5.2 Syntactic Sugar

Some expressions make a language's syntax more convenient and compact without actually adding expressivity: they make the language sweeter to use. We say that an expression is **syntactic sugar** and adds no expressive power if it can be reduced to another expression with only local changes. That is, without rewriting the entire body of the expression or making changes to other parts of the program. Such expressions are also referred to as being **macro-expressive**. They can naturally be implemented entirely within the parser or as **macros** in languages with reasonable macro systems.

For example, in Java any `FOR` statement, which has the form:

`for` (*init* ; *test* ; *incr*) *body*

can be rewritten as a WHILE statement of the form:

init ; `while` (*test*) { *body*   *incr* ; }

`FOR` therefore does not add expressivity to the language and is syntactic sugar. In Scheme, `LET` adds no power over `LAMBDA`, and `LET*` adds no power over `LET`. Java exceptions are an example of an expressive form that cannot be eliminated without completely rewriting programs in the language.

We can express the Java `FOR` loop's **reduction** in more formal notation as:

$$
\begin{gathered}
\texttt{for}\ \texttt{(}\ \textit{exp}_{\textit{init}}\ \texttt{;}\ \textit{exp}_{\textit{test}}\ \texttt{;}\ \textit{exp}_{\textit{incr}}\ \texttt{)}\ \textit{exp}_{\textit{body}} \\
\Rightarrow \\
\textit{exp}_{\textit{init}}\ \texttt{;}\ \texttt{while}\ \texttt{(}\ \textit{exp}_{\textit{test}}\ \texttt{)}\ \{\ \textit{exp}_{\textit{body}}\ \textit{exp}_{\textit{incr}}\ \texttt{;}\ \}
\end{gathered}
\tag{4}
$$

Here the pattern on the left may be reduced to the simpler form on the left during evaluation. This is an example of a general mechanism for ascribing formal semantics to syntax that is described further in the following section.

6 Semantics

6.1 Operational Semantics

An **operational semantics** is a mathematical representation of the semantics of a language. It defines how expressions are reduced to other expressions by applying a set of rules. Equivalently, it gives a set of **progress rules** for progressing from complex expressions to simpler ones, and eventually to values. Each rule has preconditions: to be applied, the rule must match the pattern of the expression. Any rule whose preconditions are met can be applied. When no rule applies the program halts. If it halts with a value, that is the result of the computation. If the semantics are self-consistent and rule expansion halts with an expression that is not a value, that indicates an error in the target program. The nature and location of the error are determined by the remaining expression and the last rule applied.

Note that the semantics influence but ultimately do not imply the implementation. For example, an operation that requires $O(n^2)$ rule applications may require only $O(n)$ operations on an actual computer. Likewise, eager and lazy substitution are often interchangeable at the semantic level. We say that two implementations are **semantically equivalent** if they always reduce identical expressions to identical vales, even if they use different underlying algorithms to implement the semantics.

The rules are expressed using a notation for reductions, conditional reductions, and substitutions. The most general form of a rule is:

$$x \Rightarrow y \tag{5}$$

"Expressions matching *x* reduce to *y*"

where *x* is a placeholder in this notation, not a variable in the programming language. These placeholders will be filled by terms, or by term variables such as $exp_{subscript}$ and $val_{subscript}$. Here, the name of the variable indicates its domain (often, expression or value), and the subscript is a tag to distinguish multiple variables in the same statement[6] Sometimes rules are written more casually using the subscripts as the variables and ignoring the domains, when the domain is irrelevant.

A specific example of a reduction rule is the additive identity in arithmetic:

$$exp_x \; \texttt{+} \; 0 \Rightarrow exp_x \tag{6}$$

We can specify general addition by:

$$num_x \; \texttt{+} \; num_y \Rightarrow \widehat{num_x + num_y} \tag{7}$$

The addition sign on the right of the reduction indicates actual addition of value; the one on the left denotes syntax for an expression. Variables num_x and num_y are expressions; the hat on the right side indicates that we mean the value corresponding to that operation.

To make this more concrete, we could give a very specific reduction:

$$\texttt{1} \; \texttt{+} \; \texttt{2} \Rightarrow \hat{3} \tag{8}$$

Here, 1 and 2 on the left of the arrow are syntax for literals, i.e., they are **numerals**. The hat on the 3 to the right of the arrow indicates that it is a value, i.e., it represents an actual **number** and not just the syntax for a number. See [Kri07, 231] for further discussion of this hat notation. We will see some cases in which the line between the expression domain and the value domain is blurry. In those, the hat notation is unnecessary.

Sometimes we have complicated conditions on a rule that constrain the pattern on the left side of ⇒. These are notated with a conditional statement the form:

$$\frac{a}{b} \tag{9}$$

"If mathematical statement *a* is true, then statement *b* is true (applicable)."

In gen-

[6]This is admittedly a suboptimal notation, since the "name" that carries the meaning for the reader but is buried in the subscript, while the "type" dominates. However, it is standard in the field.

eral, both *a* and *b* are reductions. Furthermore, there may be multiple conditions in *a*, notationally separated by spaces, that must all be true for the reduction(s) in *b* to be applied. These rules are useful for making progress when no other rule directly applies. For example, to evaluate the mathematical expression $1+(7+2)$, we must first resolve $(7+2)$. The rule for making progress on addition with nested subexpression on the right of the plus sign is:

$$\frac{exp_2 \Rightarrow num_2}{exp_1 \texttt{ + } exp_2 \Rightarrow exp_1 \texttt{ + } num_2} \tag{10}$$

which reads, "if expression #2 can be reduced to some number (by some other rules), then the entire sum can be reduced to the sum of expression #1 and that number." We of course need some way of reducing expressions on the *left* of the plus sign as well:

$$\frac{exp_1 \Rightarrow num_1}{exp_1 \texttt{ + } exp_2 \Rightarrow num_1 \texttt{ + } exp_2} \tag{11}$$

Applying combinations of these rules allows us to simplify arbitrarily nested additions to simple number additions.

There are two major interpreter designs: substitution and evaluation interpreters. Substitution interpreters transform expressions to expressions and terminate when the final expression is a literal. Their value domain is their literal domain. Evaluation/environment interpreters are implemented with an `EVAL` procedure that reduces expressions directly to values. Such a procedure recursively invokes itself, but program execution involves a single top-level `EVAL` call. These two models of implementation correspond to two styles of semantics:

1. **Small-step** operational semantics rules reduce expressions to **expressions** (like a substitution interpreter)
2. **Big-step** operational semantics rules reduce expressions to **values** (like an `EVAL` interpreter)

The following two subsections demonstrate how semantics are assigned in each. We use the context of a simple language that contains only single-argument procedure definition, variable, conditional, application, and booleans. Let these have the semantics of the equivalent forms in the Scheme language:

$$\begin{aligned} exp \quad ::= \quad & \texttt{(}\ \lambda\ \texttt{(}\ id\ \texttt{)}\ exp\ \texttt{)} \mid \\ & id \mid \\ & \texttt{(}\ if\ exp\ exp\ exp\ \texttt{)} \mid \\ & \texttt{(}\ exp\ exp\ \texttt{)} \mid \\ & \texttt{true} \mid \texttt{false} \end{aligned} \tag{12}$$

6.2 Small-step Example

6.2.1 Rules

Small-step operational semantics rules reduce expressions to expressions. Although the choice of implementation is not constrained by the style of semantics (much), small step maps most directly to a substitution-based interpreter (e.g., [Kri07, 15]). Under small-step semantics the value domain is merely the terminal subset of the expression domain, plus procedure values. That is, literals are values. It is useful to us later to define a subdomains in this definition:

$$\begin{aligned} ok \quad &::= \quad \texttt{true} \mid \texttt{(}\ \lambda\ \texttt{(}\ id\ \texttt{)}\ exp\ \texttt{)} \\ val \quad &::= \quad \texttt{false} \mid ok \end{aligned} \tag{13}$$

Those expressions require no progress rules because they are values. Variable expressions require no progress rules because variables are always substituted away by applications. Only conditional and application need be defined. Conditionals naturally have two obvious rules, that I name E-IfOk and E-IfFalse (the "E" stands for "Evaluate"):

$$\text{E-IfOk:} \quad (\ \texttt{if}\ ok\ exp_{then}\ exp_{else}\ ) \Rightarrow exp_{then} \tag{14}$$

$$\text{E-IfFalse:} \quad (\ \texttt{if}\ \texttt{false}\ exp_{then}\ exp_{else}\ ) \Rightarrow exp_{else} \tag{15}$$

Had Scheme's semantics dictated that the test expression must be a boolean, the first rule would have replaced ok with `true`, which is likely what you first expected to see there. However, recall that Scheme treats any non-false value as true for the purpose of an `IF` expression.

A perhaps less obvious rule, E-IfProgress, is required as well to complete the semantics of `IF`. When the test expression for the conditional is not in the value domain, we need some way of making progress on reducing the expression.

$$\text{E-IfProgress:} \frac{exp_{test} \Rightarrow val_{test}}{(\ \texttt{if}\ exp_{test}\ exp_{then}\ exp_{else}\ ) \Rightarrow (\ \texttt{if}\ val_{test}\ exp_{then}\ exp_{else}\ )} \tag{16}$$

Application requires a notation for expressing variable substitution. This is:

$$[id \rightarrowtail v]\,body \tag{17}$$

"Substitute v for id in $body$".

The *body* is the expression in which all instances of variable named *id* are to be replaced with the expression v. In an eager language, the v expression must be in the value domain. In a lazy language, v can be any expression. For semantic purposes the distinction between eager and lazy is irrelevant in a language without mutation and small step semantics are almost always restricted to languages without mutation.

Using this notation we can express application as a reduction. If we choose lazy evaluation, it is:

$$\text{E-App}_{\text{lazy}}: \ (\ (\ \lambda\ (\ id\ )\ exp_{body}\ )\ \ exp_{arg}\ ) \ \Rightarrow\ [id \rightarrowtail exp_{arg}]\,exp_{body} \tag{18}$$

plus a progress rule:

$$\text{E-AppProgress1}: \ \frac{exp_1 \ \Rightarrow\ val_1}{(\ exp_1\ exp_2\ ) \ \Rightarrow\ (\ val_1\ exp_2\ )} \tag{19}$$

To specify eager evaluation, we simply require the argument to be a value:

$$\text{E-App}_{\text{eager}}: \ (\ (\ \lambda\ (\ id\ )\ exp_{body}\ )\ \ val_{arg}\ ) \ \Rightarrow\ [id \rightarrowtail val_{arg}]\,exp_{body} \tag{20}$$

and introduce another progress rule (we still require rule E-AppProgress1):

$$\text{E-AppProgress2}_{\text{eager}}: \ \frac{exp_2 \ \Rightarrow\ val_2}{(\ exp_1\ exp_2\ ) \ \Rightarrow\ (\ exp_1\ val_2\ )} \tag{21}$$

6.2.2 A Proof

Because each rule is a mathematical statement, we can prove that a complex expression reduces to a simple value by listing the rules that apply. That is, by giving a list of true statements that reduce the expression to a specific value. Operational semantics are used this way, but they are more often used as a rigorous way of specifying what an interpreter should do. The reason for exploring this proof structure is that we will later us the same structure on type judgements (another kind of rule set) to prove that an expression has some interesting property, rather than a specific value.

We list the statements in the order they are applied. When reaching a conditional we must prove its antecedants. To do this, we replace each antecedant with its own proof; that is, we start nesting the conditional statements until all are satisfied.

Theorem 2. *`(if ((λ (x) x) true) false true)` reduces to `false`.*

Proof.
$$\frac{\texttt{((}\lambda\texttt{ (x) x) true)} \overset{\text{(E-App)}}{\Rightarrow} \left[\texttt{x} \rightarrowtail \texttt{true}\right]\texttt{x} \overset{\text{(subst.)}}{\Rightarrow} \texttt{true}}{\texttt{(if ((}\lambda\texttt{ (x) x) true) false true)} \Rightarrow \texttt{false}}\ \text{(E-IfTrue)}$$

□

6.3 Big-step Example

6.3.1 Rules

Under big-step operational semantics, the left side of the progress arrow is always an expression and the right side of the arrow is always a value. Thus, each rule takes a "big step" to the final value of an expression. The value and expression domains must be disjoint to make this distinction, so we define the value domain more carefully here. For our simple language from eq. 12, the big-step value domain is:

$$\begin{aligned} proc &::= \langle id, exp, \mathscr{E}\rangle \\ ok &::= proc \mid \hat{t} \\ val &::= ok \mid \hat{f} \end{aligned} \qquad (22)$$

The angle brackets in the procedure notation $\langle id, exp, \mathscr{E}\rangle$ denote a **tuple**, in this case, a 3-tuple. That is simply a mathematical vector of values, or equivalently, a list in an interpreter implementation. The specific tuple we're defining here is a 1-argument closure value, which has the expected three parts: formal parameter identifier, body expression, and captured environment. Big step operational semantics use environments in the same way that interpreters do for representing deferred substitutions. The environment is placed on the left side of the progress arrow and separated by an expression by a comma.

Now that the value domain is disjoint from the expression domain, we need some trivial rules for reducing literal and LAMBDA expressions to their corresponding values.

Forget all of the previous small-step rules. We begin big step with:

$$\begin{aligned} \text{E-True}: &\quad \texttt{true}, \mathscr{E} \Rightarrow \hat{t} \\ \text{E-False}: &\quad \texttt{false}, \mathscr{E} \Rightarrow \hat{f} \\ \text{E-Lambda}: &\quad \texttt{(}\ \lambda\ \texttt{(}\ id\ \texttt{)}\ exp\ \texttt{)}, \mathscr{E} \Rightarrow \langle id, exp, \mathscr{E}\rangle \end{aligned} \qquad (23)$$

Note that LAMBDA captures the identifier and environment, and saves and delays the expression, and that the literals ignore the environment in which they are reduced.

The rules for evaluating IF expressions are largely the same as for small step, but we can ignore the progress rules because they are implicit in the big steps:

$$\text{E-IfOk:}\quad \frac{\neg\left(exp_t, \mathscr{E} \Rightarrow \hat{f}\right) \qquad exp_c, \mathscr{E} \Rightarrow val_c}{\texttt{(}\ \texttt{if}\ exp_t\ exp_c\ exp_a\ \texttt{)}, \mathscr{E} \Rightarrow val_c}$$

$$\text{E-IfFalse:}\quad \frac{exp_t, \mathscr{E} \Rightarrow \hat{f} \qquad exp_a, \mathscr{E} \Rightarrow val_a}{\texttt{(}\ \texttt{if}\ exp_t\ exp_c\ exp_a\ \texttt{)}, \mathscr{E} \Rightarrow val_a} \qquad (24)$$

Each reduction, whether a conditional or a simple $a \Rightarrow b$, is a mathematical **statement**. Statements are either true or false[7] in the mathematical sense. Each progress rule is really a step within a proof whose theorem is

[7]...although the function to evaluate the truth value of a statement may be non-computable, as in the case of the Halting Problem.

"this program reduces to (whatever the final value is)." The ¬ operator negates the truth value of a statement. Thus the E-IfOk rule has as an antecedent "the test expression does not reduce to $\hat{f}$". We need to express it this way because our desired semantics follow Scheme's, which allow any value other than $\hat{f}$ to act like "true" for the purpose of an `IF` expression.

To express the semantics of `APP` we need a notation for extending an environment with a new binding. Since environments and substitutions are not used simultaneously, the substitution notation is repurposed to denote extending an environment:

$$\mathscr{E}\,[id \leftarrowtail val] \tag{25}$$

"Environment $\mathscr{E}$ extended with *id* bound to *val*".

Note that the arrow inside the brackets points in the opposite direction as for substitution, following the convention of [Kri07, 223]. The application rule for our toy language under big step semantics is:

$$\text{E-App}: \frac{exp_p, \mathscr{E}_1 \Rightarrow \langle id, exp_b, \mathscr{E}_b \rangle \quad exp_a, \mathscr{E}_1 \Rightarrow val_a \quad exp_b, \mathscr{E}_b\,[id \leftarrowtail val_a] \Rightarrow val_b}{\texttt{(}\ exp_p\ \ exp_a\ \texttt{)}\ , \mathscr{E}_1 \Rightarrow val_b} \tag{26}$$

This reads,

"**if**(expression exp_p reduces to a procedure in the current environment ($\mathscr{E}_1$),

and argument expression exp_a reduces to value in the current environment, and

and the body of that procedure evaluated in the procedure's stored environment ($\mathscr{E}_b$) extended with *id* bound to the argument's value reduces to value val_b,)

then the application of the procedure expression to the argument expression in the current environment is val_b."

Observe that the body expression is evaluated in the stored environment extended with the new binding, creating lexical scoping. Were we to accidentally use the current environment there we would have created dynamic scope. We conclude with the trivial variable rule:

$$\text{E-Var}: id, \mathscr{E}\,[id \leftarrowtail val] \Rightarrow val \tag{27}$$

Examining our big-step rules, we see that they map one-to-one to the implementation of an interpreter for the language. Each rule is one case inside the `EVAL` procedure, or inside the parser for ones that we choose to rewrite at parse time. Within a rule, each antecedant corresponds to one recursive call to `EVAL`. For example in the E-App rule, there are three antecedants. These correspond to the recursive calls to evaluate the first subexpression (which should evaluate to a procedure), the second (i.e., the argument), and then the body. That last call to evaluate the body is usually burried inside `APPLY`. The environments specified on the left sides of the antecedants tell us which environments to pass to `EVAL`.

See [Kri07] chapter 23 for an excellent set of examples of progress rules for big-step operational semantics.

6.3.2 A Proof

This is a proof of the same statement from the small-step example, now proven with the big-step semantics. Because the nesting gets too deep to fit the page width, I created a separate lemma for the conditional portion of the proof.

Lemma 1. *`((λ (x) x) true)`, $\mathscr{E}$ reduces to $\hat{t}$*

Proof.

$$\frac{\texttt{(λ (x) x)}\ , \mathscr{E} \overset{\text{(E-Lambda)}}{\Rightarrow} \langle\ \texttt{x}\ ,\ \texttt{x}\ , \mathscr{E}\rangle \qquad \texttt{true}\ , \mathscr{E} \overset{\text{(E-True)}}{\Rightarrow} \hat{t} \qquad \texttt{x}\ , [\ \texttt{x} \leftarrowtail \hat{t}\ ]\mathscr{E} \overset{\text{(E-Var)}}{\Rightarrow} \hat{t}}{\texttt{((λ (x) x) true)}\ , \mathscr{E} \Rightarrow \hat{t}}\text{(E-App)} \quad \square$$

Theorem 3. *(if ((λ (x) x) true) false true)*, $\mathscr{E}$ *reduces to* $\hat{f}$.

Proof.

1. Because `((λ (x) x) true)` $,\mathscr{E} \overset{\text{(Lemma 1)}}{\Rightarrow} \hat{t}$, and $\hat{t} \neq \hat{f}$ the negation of the contradiction of Lemma 1, $\neg\left(\texttt{((λ (x) x) true)}\,,\mathscr{E} \Rightarrow \hat{f}\right)$, is also true.

2. $$\frac{\neg\left(\texttt{((λ (x) x) true)}\,,\mathscr{E} \Rightarrow \hat{f}\right) \qquad \texttt{false}\,,\mathscr{E} \Rightarrow \hat{f}}{\texttt{(if ((λ (x) x) true) false true)}\,,\mathscr{E} \Rightarrow \hat{f}}\text{E-IfOk}$$

□

7 The λ Calculus

The λ calculus is Church's [Chu32] minimalist functional model of computation. Church showed that all other programming constructs can be eliminated by reducing them to single-argument procedure definition (i.e., abstraction; lambda), variables, and procedure application. Variations of λ calculus are heavily used in programming language research as a vehicle for proofs. Outside research, there are several motivations for studying λ calculus and reductions to it from more complex languages.

Philosophically, λ calculus is the[8] foundation for our understanding of computation and highlights the power of abstraction. Practically, understanding the language and how to reduce others to it changes the way that one thinks about (and applies) constructs in other languages. This leads the way to emulating constructs that are missing in a language at hand, which makes for a better programmer. For example, Java lacks lambda. The Java API designers quickly learned to use anonymous classes to create anonymous closures, enabling the use of first-class function-like objects in a language that does not support functions. C++ programmers discovered a way to use the polymorphic mechanism of templates as a complete macro language.

On learning a new language, the sophisticated programmer does not learn the specific forms of that language blindly but instead asks, "which forms create closures, recursive bindings, iteration, etc. in this language?". If any of the desired features are missing, that programmer then emulates them, using techniques learned by emulating complex features in the minimalist λ calculus. So, although implementing Church Booleans is just an academic puzzle for most programmers, that kind of thought process is valuable in implementing practical applications.

André van Meulebrouck describes an alternative motivation:

> "Perhaps you might think of Alonzo Church's λ-calculus (and numerals) as impractical mental gymnastics, but consider: many times in the past, seemingly impractical theories became the underpinnings of future technologies (for instance: Boolean Algebra [*i.e., today's computers that operate in binary build massive abstractions using only Boole's theoretical logic!*]).
>
> Perhaps the reader can imagine a future much brighter and more enlightened than today. For instance, imagine computer architectures that run combinators or λ-calculus as their machine instruction sets."[9]

[8]or at least, one of the two...

[9]http://www.mactech.com:16080/articles/mactech/Vol.07/07.06/ChurchNumerals/

7.1 Syntax

The λ calculus is a language with surprisingly few primitives in the expression domain[10]:

$$\begin{aligned} var &::= id \\ abs &::= \texttt{λ}\ id\ \texttt{.}\ exp \\ app &::= exp\ exp \\ exp &::= var \mid abs \mid app \mid \texttt{(}\ exp\ \texttt{)} \end{aligned}$$

The last expression on the right simply states that parentheses may be used for grouping.

The language contains single value type, the single-argument procedure, in the value domain. In set notation this is:

$$val = proc = var \times exp$$

and in BNF:

$$val \quad ::= \quad \texttt{λ}\ id\ \texttt{.}\ exp$$

The abbreviated names used here and in the following discussions are mnemonics for: 'id' = 'identifier', 'abs' = 'abstraction' (since λ creates a procedure, which is an abstraction of computation), 'app' = 'procedure application', 'exp' = 'expression', 'proc' = 'procedure', and 'val' = 'value'.

7.2 Semantics

The formal semantics are simply those of substitution [Pie02, 72]:

App-Part 1: (reduce the procedure expression towards a value)

$$\frac{exp_p \Rightarrow exp'_p}{exp_p exp_a \Rightarrow exp'_p exp_a} \tag{28}$$

App-Part 2: (reduce the actual parameter towards a value)

$$\frac{exp_a \Rightarrow exp'_a}{exp_p exp_a \Rightarrow exp_p exp'_a} \tag{29}$$

App-Abs: (apply a procedure to a value)

$$\texttt{λ}\ id\ \texttt{.}\ exp_{body}\ \ val \ \ \Rightarrow\ \ [id \rightarrowtail val]\, exp_{body} \tag{30}$$

The App-Abs rule relies on the same syntax for the *val* value and *abs* expression, which is fine in λ calculus because we're using pure textural substitution. In the context of a true value domain that is distinct from the expression domain, we could express it as an *abs* evaluation rule for reducing a procedure expression to a procedure value and and an application rule written something like:

$$\frac{val_p = \texttt{λ}\ id\ \texttt{.}\ exp_{body}}{val_p\ val_a \Rightarrow\ [id \rightarrowtail val_a]\, exp_{body}} \tag{31}$$

7.3 Examples

For the sake of giving simple examples, momentarily expand λ calculus with the usual infix arithmetic operations and integers. Consider the evaluation of the following expression:

$$\lambda\ x\ .\ (x+3)\ 7 \tag{32}$$

[10]This is specifically a definition of the *untyped* λ-calculus.

This is an app expression. The left sub-expression is a abs, the right sub-expression is an integer (7). Rule App-Abs is the only rule that applies here, since both the procedure and the integer cannot be reduced further. App-Abs replaces all instances of x in the body expression $(x+3)$ with the right argument expression, 7:

$$\begin{aligned}
&\lambda\, x\,.\,(x+3)\;7 && (33)\\
&\quad\Rightarrow\;[x \rightarrowtail 7](x+3) && (34)\\
&\quad=(7+3) && (35)\\
&\quad=10 && (36)
\end{aligned}$$

Arithmetic then reduces this to 10.

Now consider using a procedure as an argument:

$$(\lambda f\,.\,(f\;3))\;(\lambda x.(1+x)) \qquad (37)$$

This is again an app. In this case, both the left and right are abs expressions. Applying rule App-Abs substitutes the right expression for f in the body of the left procedure, and then we apply App-Abs again:

$$\begin{aligned}
&(\lambda f\,.\,(f\;3))\;(\lambda x.(1+x)) && (38)\\
&\Rightarrow [\;f \;\rightarrowtail\; (\lambda x.(1+x))\;](f\;3)) && (39)\\
&= \lambda x.(1+x)\;3 && (40)\\
&\Rightarrow [x \rightarrowtail 3](1+x) && (41)\\
&= (1+3) && (42)\\
&= 4 && (43)
\end{aligned}$$

7.4 Partial Function Evaluation

Because procedure application is left associative and requires no function application operator:

$$f\;x\;y = (f\;x)\;y \qquad (44)$$

we can emulate the functionality of multiple-argument procedures using single argument procedures. For example, the "two-argument" procedure that sums its arguments is:

$$\lambda x.\lambda y.(x+y) \qquad (45)$$

The outer expression is an abs (procedure definition), whose body is another abs. This is a first-order procedure. When placed into nested app expressions, the procedure returns another procedure, which then consumes the second app's argument:

$$\begin{aligned}
&(\lambda x.\lambda y.(x+y))\;\;1\;\;2 && (46)\\
&\quad\Rightarrow \lambda y.(1+y)\;\;2 && (47)\\
&\quad\Rightarrow 1+2 && (48)\\
&\quad= 3 && (49)
\end{aligned}$$

The process of converting a 0th-order function of n arguments into an nth-order function of one argument, like the one above, is called **currying**. It is named for the logician Haskell Curry, who was not its inventor[11].

When an nth-order function is applied to $k < n$ arguments, in λ calculus, the result reduces to an $(n-k)$th order function. The resulting function "remembers" the arguments that have been provided because they have already been substituted, and it will complete the computation when later applied to the remaining arguments. This is called **partial function evaluation**, and is a feature of some languages including Haskell (which is *also* named for Haskell Curry, who did not invent it either.) For example, the addition function above can be partially evaluated to create an "add 5" function:

$$\begin{aligned}
&(\lambda x.\lambda y.(x+y))\;\;5 && (50)\\
&\quad\Rightarrow \lambda y.(5+y) && (51)
\end{aligned}$$

[11]The idea is commonly credited to Schönfinkel in the 20th century, and was familiar to Frege and Cantor in the 19th [Pie02, 73]

7.5 Creating Recursion

A **fixed point** of a function f is a value v such that $f(v) = v$ (in mathematical notation; in λ calculus, we would say $f\ v = v$. A function may have zero or more fixed points. For example, the identity function $\lambda x.x$ has infinitely many fixed points. Let $s = \lambda x.x^2$ be the function that squares its argument; it has fixed points at 0 and 1.

A **fixed point combinator** is a function that computes a fixed point of another function. This is interesting because it is related to recursion. Consider the problem of defining a recursive function in λ calculus. For example, define factorial (for convenience, we temporarily extend the language with conditionals and integers; those can be reduced as shown previously):

$$\begin{array}{l} \lambda n.(\text{if } (\text{iszero } n) \\ \qquad 1 \\ \qquad (n * (f\ (n-1)))) \end{array}$$

The problem with this definition is that we need the f embedded inside the recursive case to be bound to the function itself, but that value does not exist at the time that the function is being defined. Alternatively, the problem is that f is a free variable. Adding another abs expression captures f:

$$\begin{array}{l} \boxed{\lambda f.}\ \lambda n.(\text{if } (\text{iszero } n) \\ \qquad 1 \\ \qquad (n * (f\ (n-1)))) \end{array}$$

This just says that if we already had the factorial function that operated on values less than n, we could implement the factorial function for n. That's close to the idea of recursion, but is not fully recursive because we've only implemented one inductive step. We need to handle all larger values. To let this inductive step run further, say that f is the function that we're defining, which means that the inner call requires two arguments: f and $n-1$:

$$\begin{array}{l} \lambda f.\lambda n.(\text{if } (\text{iszero } n) \\ \qquad 1 \\ \qquad (n * (f\ \boxed{f}\ (n-1)))) \end{array}$$

Call this entire function g. It is a function that, given a factorial function, creates the factorial function. At first this does not sound useful–if we had the factorial function, we wouldn't need to write it! However, consider that what we have defined g such that $(gf) = f$...in other words, the factorial function is the fixed point of g. For this particular function, we can find the fixed point by binding it and then calling it on itself. Binding and applying values are accomplished using abs and app expressions. An expression of the form:

$$(\lambda z.z\ z)\ g\ \Rightarrow\ g\ g \qquad (52)$$

applies g to itself. Wrapping our entire definition with this:

$$\begin{array}{l} \boxed{(\lambda z.z\ z)} \\ \quad (\lambda f.\lambda n.(\text{if } (\text{iszero } n) \\ \qquad\quad 1 \\ \qquad\quad (n * (f\ f\ (n-1))))) \end{array}$$

produces a function that is indeed factorial, albeit written in a strange manner. Convince yourself of this by running it in Scheme, using the following translation and application to 4:

```
(
 ((lambda (z) (z z))
  (lambda (f)
    (lambda (n)
      (if (zero? n)
          1
          (* n ((f f) (- n 1)))))))

4)
```

When run, this correctly produces $4! = 4*3*2*1 = 24$.

This exercise demonstrates that it is possible to implement a recursive function without an explicit recursive binding construct like LETREC. For the factorial case, we manually constructed a generator function g and its fixed point. Using a fixed point combinator we can automatically produce such fixed points, simplifying and generalizing the process.

Curry discovered the simplest known fixed point combinator for this application. It is known as the **Y combinator** (a.k.a. applicative-order Z combinator as expressed here), and is defined as:

$$\boxed{\begin{aligned} Y \;=\; & \lambda f . \\ & ((\lambda z \,.\, z\, z) \\ & \;(\lambda x \,.\, f(\lambda y \,.\, x\, x\, y))) \end{aligned} \qquad (53)}$$

When applied to a generator function, Y finds its fixed point and produces that recursive function. The Y combinator is formulated so that the generator need not apply its argument to itself. That is, the step where we rewrote $(f\ (n-1))$ as $(f\ f\ (n-1))$ in our derivation is no longer necessary.

A Scheme implementation of Y and its use to compute factorial are below. The use of the DEFINE statement is merely to make the implementation more readable. Those can be reduced to LAMBDA and application expressions.

```
; Creates the fixed point of its argument
(define Y
  (lambda (f)
    ((lambda (z) (z z))
     (lambda (x) (f (lambda (y) ((x x) y)))))))

; Given the factorial function, returns the factorial function
(define generator
  (lambda (fact)
     (lambda (n)
       (if (zero? n)
           1
           (* n (fact (sub1 n)))))))

; The factorial function: prints (lambda...)
(Y generator)

; Example: prints 24
((Y generator) 4)
```

8 Macros

We've seen that a parser can reduce macro-expressive forms to other forms to minimize the number of cases that need to be handled in the compiler/interpreter. For example, a short-circuiting OR expression like the one in Scheme can be reduced within the parser by the small-step rule:

$$\texttt{(}\ \texttt{or}\ exp_1\ exp_2\ \texttt{)} \Rightarrow \texttt{(}\ \texttt{let}\ \texttt{(}\ \texttt{[}\ id\ exp_1\ \texttt{]}\ \texttt{)}\ \texttt{(}\ \texttt{if}\ id\ id\ exp_2\ \texttt{)}\ \texttt{)} \qquad (54)$$

Languages with a **macro** system feature allow the programmer to add rules such as this to the parser. They are effectively plugin-modules for the parser. Macros are written in a separate language that is often similar to the base language, and they generally describe pattern matching behavior. They extend the syntax of the language in ways that cannot be achieved using procedures alone. For example, a short-circuiting OR cannot be written using only procedures, IF, and application in a language with eager evaluation of procedure arguments.

In the OR example, it is important that the identifier *id* does not appear as a free variable on the other expressions. If it did, the macro would accidentally capture that variable and change the meaning of exp_1 and exp_2. A **hygienic** macro system is one in which identifiers injected into code by the macro system cannot conflict with ones already present in expressions. One way to achieve this is to append the level of evaluation at which an identifier was inserted to the end of its name. Identifiers in the original source code are at level 0, those created by first-level macro expansion are at level 1, those created by macros emitted by the first-level macros are at level 2, and so on. Not all macro systems are hygienic. While Scheme's macro system is (since R5R6) and is generally considered both clean and powerful, the most frequently used macro system–that of C/C++ –is not. This does not mean that C macros are useless, just that extreme care must be taken when using them.

8.1 C Macros

The C macro system contains two kinds of statements: `#if` and `#define`. Without defining its semantics here, an example[12] illustrates how they are typically employed:

```
-#include <stdio.h>

#if defined(_MSC_VER)
//      Windows
#       define BREAK   ::DebugBreak();
#elif defined(__i386__) && defined(__GNUC__)
//      gcc on some Intel processor
#       define BREAK __asm__ __volatile__ ( "int $3" );
#else
//      Hopefully, some other gcc
#       define BREAK ::abort()
#endif

#define ASSERT(test_expr, message) \
    if (! (test_expr)) {\
        printf("%s\nAssertion \"%s\" failed in %s at line %d.\n", \
        message, #test_expr, __FILE__, __LINE__);\
        BREAK;\
    }

int main(int argc, char** argv) {
     int x = 3;
     ASSERT(x > 4, "Something bad");
     return 0;
}
```

The `#if` statements are used to determine, based on expressions including variables such as `_MSC_VER` that are defined at compile time, what compiler and operating system the code is being compiled for. They allow the program to function differently on different machines without the expense of a run-time check. Furthermore, certain statements that are not legal on one compiler can be avoided entirely, such as the inline assembly syntax used for gcc. The `#define` statement creates a new macro. By convention, macros are given all uppercase names in C. Here, two macros are defined: `BREAK`, which halts execution of the program when the code it emits is invoked, and `ASSERT`, which conditionally halts execution if a test expression returns false. `ASSERT` cannot be a procedure for two reasons: first, it would be nice to define it so that in an optimized build the assertions are removed (not shown here), and second, because we do not want to evaluate the message expression if the test passes.

Within the body of the `ASSERT` definition we see several techniques that are typical of macro usage. The special variables `__FILE__` and `__LINE__` indicate the location at which the macro was invoked. Unlike procedures in most languages, macros have access to the source code context from which they were invoked. This allows them to customize error reporting behavior. The expression `#test_expr` is applying the `#` operator to the `test_expr` macro variable. This operator quotes the source code, converting it from code into a string that may then be printed. Procedures have no way of accessing the expressions that produced their arguments, let alone the source code for those expressions. Note that where it is used as the conditional for `if`, `test_expr` is wrapped in parentheses. This is necessary because C macros operate at a pre-parse (in fact, pre-tokenizer!) level, unlike Scheme macros. Without these extra parentheses, the application of the not operator `(!)` might be parsed differently depending on the operator precedence of other operators inside the expression. This is

[12]Adapted from the G3D source code, http://g3d-cpp.sf.net.

generally considered a poor design decision of the C language, not a feature, although it can be exploited in useful ways to create tokens at compile time.

C's macro system is practical, though ugly. The C++ language addresses many of its shortcomings by introducing two other language features for creating new syntax: **templates** and **operator overloading**. Templates were originally introduced as a straightforward polymorphic type mechanism, but have since been exploited by programmers as a general metaprogramming mechanism that has Turing-equivalent computational power.

9 Type Systems

9.1 Types

A **type** is any property of a program that we might like to check, regardless of whether it is computable. Examples of such properties are "Function f returns an even integer", "Program P halts on input x", and "Program P never dereferences a NULL pointer". We will restrict ourselves to properties of values, which is what is commonly meant by programers when they refer to "types". In this context, a type is any of the following equivalent representations:

Definition	**Example**
Set of values	$x = \{0, 1, 2, ...\}$
Specific subset of the value domain	$uint = \{x \in num \subseteq val \mid x \geq 0\}$
Condition	$x \in num$ and $x \geq 0$
Predicate	`(define (uint?  x) (and (number?  x) (>= x 0)))`
Precise description	non-negative integers

Values have types in all programming languages. This is necessary to define the semantics of operations on the values. In some implementations of some languages, types are explicitly stored with the values at runtime. This is the case for Java Objects and Python and Scheme values, as well as C++ classes under most compilers. In other implementations and languages, types are implicit in the code produced by a compiler. This is the case in C. For an implementation to avoid storing explicit type values at runtime, it must ascribe **static types** to variables, meaning that each variable can be bound to values of a single type that is determined at compile time. Languages that retain run-time type information may have either static or **dynamic types** for variables. Dynamically typed variables can be bound to any type of value. Few languages allow the type of a *value* to change at runtime. One exception is C++, where the `reinterpret_cast` operator allows the program to reinterpret the bits stored in memory of a value as if they were a value of another type.

9.2 Type Checking

Recall that an operational semantics is a set of rules for reducing expressions to *values*. Although they are generally employed as a formal method for specifying how to write an interpreter or compiler, those rules can be used to prove that a specific expression evaluates to a specific value. **Type judgements** are a parallel set of rules for reducing expressions to *types*. They are used to prove that an expression must evaluate to a specific type. An interpreter would encounter an error condition under operational semantics if it was unable to apply any rule to reduce an expression further. Likewise, a **type checker** encounters an error condition under a set of type judgements if it is unable to apply any rule to reduce an expression further. In both cases, that situation indicates an error in the program being checked.

Types are more general than values, so proving that an expression reduces to some value in a type is less powerful than proving that it reduces to a specific value. However, proving that it reduces to a value is exactly as hard as executing the program, which means that it is undecidable in general. It is easier to prove that an expression reduces to some value in a type, without determining the exact value. Furthermore, this proof can be undertaken in the form of a derivation, where the result type and a proof of its correctness are discovered simultaneously. This process is called **type checking**.

If the types and judgements are carefully defined, type checking is decidable. That means we can guarantee that the type checker will terminate. Because they are just rules, the type judgements are a kind of meta-language. This is an example of intentionally designing a language to be less computationally powerful in order to avoid the incompleteness theorem.

A **type system** comprises the types, the type judgments, and the algorithm for applying the type judgments to form a proof/discover the result type. Note that type systems reduce the expressivity of a programming language by restricting the kinds of expressions that are legal, beyond the restrictions imposed by the grammar itself. This is desirable because a well-defined type system generally prohibits programs that would have been incorrect anyway. Of course, we always run the risk that some correct programs will be excluded by this choice. Furthermore, when we limit the type system in order to make type checking decidable, we lose the ability to eliminate *all* incorrect programs. Therefore, a decidable type system can only guarantee that a program is **valid**, and not **correct**.

9.3 Type Judgment Notation

Consider a simple language:

$$\begin{aligned}
\textit{bool-lit} &::= \texttt{true} \mid \texttt{false} \\
\textit{lit} &::= \textit{num-lit} \mid \textit{bool-lit} \\
\textit{and} &::= \texttt{(}\ \texttt{and}\ \textit{exp}\ \textit{exp}\ \texttt{)} \\
\textit{equal} &::= \texttt{(}\ \texttt{equal}\ \textit{exp}\ \textit{exp}\ \texttt{)} \\
\textit{plus} &::= \texttt{(}\ \texttt{+}\ \textit{exp}\ \textit{exp}\ \texttt{)} \\
\textit{if} &::= \texttt{(}\ \texttt{if}\ \textit{exp}\ \textit{exp}\ \textit{exp}\ \texttt{)} \\
\textit{exp} &::= \textit{lit} \mid \textit{and} \mid \textit{equal} \mid \textit{plus} \mid \textit{if}
\end{aligned}$$

$$\begin{aligned}
\textit{bool} &::= \hat{t} \mid \hat{f} \\
\textit{val} &::= \textit{num} \mid \textit{bool}
\end{aligned} \tag{55}$$

We could use the operational semantics notation to express type judgements, e.g.,

$$\text{T-and:}\ \frac{\textit{exp}_1, \mathscr{E} \Rightarrow \textit{bool} \quad \textit{exp}_2, \mathscr{E} \Rightarrow \textit{bool}}{\texttt{(}\ \texttt{and}\ \textit{exp}_1\ \textit{exp}_2\ \texttt{)}, \mathscr{E} \Rightarrow \textit{bool}} \tag{56}$$

However, there would be two drawbacks to that approach. First, it looks like an operational semantics, so we might be confused as to what we were seeing. Second, environments don't exist during static evaluation (i.e., at "compile time"), so $\mathscr{E}$ is not useful. That is, environments store values that we've already discovered, but for type checking we aren't discovering values. We need a different kind of environment for type-checking: one that stores types we've already discovered. Therefore, type judgements use a slightly different notation than operational semantics. The above statement is written as:

$$\text{T-and :}\ \frac{\Gamma \vdash \textit{exp}_1\ :\ \textit{bool} \quad \Gamma \vdash \textit{exp}_2\ :\ \textit{bool}}{\Gamma \vdash \texttt{(}\ \texttt{and}\ \textit{exp}_1\ \textit{exp}_2\ \texttt{)}\ :\ \textit{bool}} \tag{57}$$

This reads, "if gamma proves that expression 1 has type bool and gamma proves that expression 2 has type bool, then gamma proves that (and expression 1 expression 2) has type bool". Gamma (Γ) is a variable representing a **type environment**. Just as the root value environment contains the library functions, the root type environment contains the types of literals and the library functions. It is then extended with other types that are discovered. The turnstile[13] operator "$\vdash$" is the part that reads "proves". It replaces the comma from operational semantics notation. The colon reads "has type"; it replaces the arrow from operational semantics (except for the colon right next to "T-and"; that's just a natural language colon indicating the name of the judgement). The T-plus and T-equal judgments are similar to T-and, but with the appropriate types inserted.

We need a set of trivial judgements for the literals:

$$\begin{aligned}
\text{T-num:}\ & \textit{num-lit}\ :\ \textit{num} \\
\text{T-bool:}\ & \textit{bool-lit}\ :\ \textit{bool}
\end{aligned} \tag{58}$$

[13] `\vdash` in Latex

All that remains is `IF`. The type of a conditional depends on what the types of the consequent and alternate expressions are. Since we don't statically know which branch will be taken, we are unable to ascribe a single type to an `IF` expression if it allows those expressions to have different types as Scheme does. This will limit the number of invalid programs that we can detect. So let us choose to define our language in the same manner as C and Java, where both branches of the `IF` must have the same type, but there is no restriction on what that type must be:

$$\text{T-if}: \quad \frac{\Gamma \vdash exp_t \ : \ bool \quad \Gamma \vdash exp_c \ : \ \tau \quad \Gamma \vdash exp_a \ : \ \tau}{\Gamma \vdash \texttt{(} \ \texttt{if} \ exp_t \ exp_c \ exp_a \ \texttt{)} \ : \ \tau} \tag{59}$$

Given type judgments for all of the expressions in our language, we can now discover/prove the type of any expression. For example, the following is a proof that `(if (and true false) 1 4)` has type *bool*:

$$\dfrac{\dfrac{\Gamma\vdash\texttt{true}\overset{\text{(T-bool)}}{:}bool \quad \Gamma\vdash\texttt{false}\overset{\text{(T-bool)}}{:}bool}{\Gamma\vdash\texttt{(and true false)}:bool}\,\text{(T-and)} \quad \Gamma\vdash\texttt{1}\overset{\text{(T-num)}}{:}num \quad \dfrac{\Gamma\vdash\texttt{4}\overset{\text{(T-num)}}{:}num \quad \Gamma\vdash\texttt{2}\overset{\text{(T-num)}}{:}num}{\Gamma\vdash\texttt{(+ 4 2)}:num}\,\text{(T-plus)}}{\Gamma\vdash\texttt{(if (and true false) 1 (+ 4 2))} : num}\,\text{(T-if)}$$

9.4 Procedure Types

Extend the little language from the previous section with procedures:

$$\begin{aligned} lambda &::= \texttt{(} \ \lambda \ \texttt{(} \ id^* \ \texttt{)} \ exp \ \texttt{)} \\ app &::= \texttt{(} \ exp^+ \ \texttt{)} \\ exp &::= \ldots \mid lambda \mid app \end{aligned}$$

$$\begin{aligned} proc &::= \langle id^*, exp, evt \rangle \\ val &::= \ldots \mid proc \end{aligned} \tag{60}$$

Recall our earlier notation for procedure types, e.g.:

$$\texttt{(}\lambda\texttt{ (x y) (+ x (if y 1 2)))} : num \times bool \rightarrow num \tag{61}$$

Here, "$\times$" denotes the Cartesian set product, which is equivalent to denoting the set of all possible tuples of values from its operand sets and the arrow indicates that this is a procedure mapping the argument types to the return type.

The application rule is straightforward. For one argument it is:

$$\text{T-app:} \quad \frac{\Gamma \vdash exp_p : (a \rightarrow r) \quad \Gamma \vdash exp_a : a}{\Gamma \vdash \texttt{(} \ exp_p \ exp_a \ \texttt{)} : r} \tag{62}$$

For multiple arguments it generalizes with $n+1$ terms in the condition:

$$\boxed{\text{T-app:} \quad \frac{\Gamma \vdash exp_p : (a_1 \times \ldots \times a_n \rightarrow r) \qquad \forall i, \Gamma \vdash exp_i : a_i}{\Gamma \vdash \texttt{(} \ exp_p \ exp_1 \ \ldots \ exp_n \ \texttt{)} : r} \qquad (63)}$$

It is difficult to write a type judgement for `LAMBDA` without additional information. Some languages, like ML, tackle that challenge using a process called type inference. For the moment, let us avoid the problem and require the programmer to specify additional information in the form of an **annotation**. This is the strategy taken by languages like Java and C++. For our annotations, change the syntax of `LAMBDA` to be:

$$\begin{aligned} type &::= \texttt{bool} \mid \texttt{num} \mid \texttt{(} \ [type] \ (\ \times \ type \ )^* \rightarrow type \ \texttt{)} \\ lambda &::= \texttt{(} \ \lambda \ \texttt{(} \ (\ id \ \texttt{:} \ type \ )^* \ \texttt{)} \ \texttt{:} \ type \ exp \ \texttt{)} \end{aligned} \tag{64}$$

An example of a correct procedure definition in this new syntax is:

$$\texttt{(}\lambda\texttt{ (x:num y:bool):bool (if (equal x 3) y false))} \tag{65}$$

Note that this is essentially the same as C++/Java syntax, except that we write types after identifiers instead of before them. Now we can express a type judgement for procedure creation:

$$\text{T-lambda:}\quad \frac{\Gamma[id_1 \leftarrow a_1, \ldots, id_n \leftarrow a_n] \vdash exp_b : r}{\Gamma \vdash \texttt{(}\ \lambda\ \texttt{(}\ id_1 \texttt{ : } a_1 \ldots id_n \texttt{ : } a_n\ \ exp_b\ \texttt{)}\ \ \texttt{:}\ r\ : (a_1 \times \ldots \times a_n \rightarrow r)} \tag{66}$$

This reads, "If gamma extended such that id_1 has type a_1, and so on for all other identifiers, proves that the body expression has type r, then gamma proves that the procedure has type $(a_1 \times \ldots \times a_n \rightarrow r)$".

T-app and T-lambda each check one half of the type contract, and together ensure that procedures are correctly typed:

- "When typing the function declaration, we *assume* the argument will have the right type and guarantee that the body, or result, will have the promised type.
- When typing a function application, we *guarantee* the argument has the type that the function demands, and *assume* the result will have the type the function promises." [Kri07, 245]

Observe that because we already have a mechanism for determining the type of an arbitrary expression the return type annotations are spurious.

From a theoretical perspective, type annotations are merely a crutch for the type checker and would not be required in any language. From the perspective of a practical programmer, they are a feature and not a drawback. The type annotations are a form of documentation. Many programmers feel that they are likely to get the annotations right and instead make mistakes in the implementation of a procedure. If this is true, then a system that infers procedure types will tend to generate misleading error messages that blame the application and not the procedure when the argument types do not match due to an error in the procedure.

9.5 Inference

The following is an alternative discussion of [Kri07] chapter 30.

Type inference solves the problem of typing `LAMBDA` and `APP` without requiring explicit annotations on `LAMBDA`. A Hindley-Milner-Damas type system operates by discovering **constraints** on the types of expressions and then **unifying** those constraints. This process is used by languages such as ML and Haskell.

The unification process corresponds to running a substitution interpreter in a metalanguage where the values are types. Say we have the following language for our programs:

$$\begin{aligned}
\textit{bool-lit} &::= \texttt{true} \mid \texttt{false} \\
\textit{num-lit} &::= \ldots \mid \texttt{-1} \mid \texttt{0} \mid \texttt{1} \mid \ldots \\
\textit{lit} &::= \textit{num-lit} \mid \textit{bool-lit} \\
\textit{app} &::= \texttt{(}\ \textit{exp}\ \textit{exp}\ \texttt{)} \\
\textit{lambda} &::= \texttt{(}\ \lambda\ \texttt{(}\ \textit{id}\ \texttt{)}\ \textit{exp}\ \texttt{)} \\
\textit{var} &::= \textit{id} \\
\textit{exp} &::= \textit{lit} \mid \textit{app} \mid \textit{lambda}\ \textit{var} \\
\\
\textit{bool} &::= \widehat{\texttt{t}} \mid \widehat{\texttt{f}} \\
\textit{num} &::= \ldots \mid \widehat{\texttt{-1}} \mid \widehat{\texttt{0}} \mid \widehat{\texttt{1}} \mid \ldots \\
\textit{proc} &::= \langle \textit{id}, \textit{exp}, \textit{evt} \rangle \\
\textit{val} &::= \textit{num} \mid \textit{bool} \mid \textit{proc}
\end{aligned} \tag{67}$$

There is a corresponding metalanguage on the types:

$$\begin{aligned}
\textit{tlit} &::= \widehat{\texttt{num}} \mid \widehat{\texttt{bool}} \\
\textit{tvar} &::= \textit{exp} \\
\textit{tproc} &::= \textit{texp} \ \rightarrow \textit{texp} \\
\textit{texp} &::= \textit{tlit} \mid \textit{tvar} \mid \textit{tproc} \\
\\
\textit{tval} &::= \textit{tlit} \mid \textit{tval} \ \rightarrow \textit{tval}
\end{aligned} \tag{68}$$

$$\tag{69}$$

Note that the metalanguage isn't quite so clear about distinguishing its expression and value domains. That's because we're going to write a substitution interpreter (i.e., no mutation!) for the metalanguage, and for a substitution interpreter the expression/value distinction is less significant. *tval* is essentially the same as *texp*, except it requires that there be no variables nested anywhere in a type.

A type inference system assigns types to all bound program variables. In the process of doing so, it also finds the types of all other expressions in the same way as a type checker. Thus it checks types while discovering them.

Within the inference system, the type variables (*tvar*) are program expressions (*exp*), because the type variables are what we want to find the type values of, and those are expressions. For example, we might describe the program:

$$\texttt{((}\lambda\texttt{ (x) x) 7)} \tag{70}$$

as having the following variables:

$$\text{Let } t_1 = [[\ \texttt{((}\lambda\texttt{ (x) x) 7)}\ ]] \tag{71}$$

$$t_2 = [[\ \texttt{(}\lambda\texttt{ (x) x)}\ ]] \tag{72}$$

$$t_3 = [[\ \texttt{7}\ ]] \tag{73}$$

$$t_4 = [[\ \texttt{x}\ ]] \tag{74}$$

where [[...]] means "the type of" the enclosed expression. In solving for these variables, we encounter the following constraints, from the application, the lambda, and the literal expressions:

$$[[\ \texttt{(}\lambda\texttt{ (x) x)}\ ]] = [[\ \texttt{7}\ ]] \rightarrow [[\ \texttt{((}\lambda\texttt{ (x) x) 7)}\ ]] \tag{75}$$

$$[[\ \texttt{(}\lambda\texttt{ (x) x)}\ ]] = [[\ \texttt{x}\ ]] \rightarrow [[\ \texttt{x}\ ]] \tag{76}$$

$$[[\ \texttt{7}\ ]] = \widehat{\texttt{num}} \tag{77}$$

Writing these in terms of the type variables, we have:

$$\begin{aligned} t_2 &= t_3 \rightarrow t_1 \\ t_2 &= t_4 \rightarrow t_4 \\ t_3 &= \widehat{\texttt{num}} \end{aligned} \tag{78}$$

This is a set of simultaneous constraints, which we can solve by substitution (much like solving simultaneous equations in algebra!) A substitution type interpreter solves for their values. This type interpreter maintains a stack of constraints and a type environment (a.k.a. a **substitution**). These are analogous to the stack of expressions and the environments in a program interpreter. The type environment initially contains the types of the built-in procedures. The stack initially contains the constraints obtained directly from the program. While the stack is not empty, the interpreter pops a constraint of the form $X = Y$ off the top, where X and Y are both $texp$, and processes that constraint as follows:

- If $X \in tvar$ and $X = Y$, do nothing. This constraint just said a type variable was equal to itself.
- If $X \in tvar$, replace all occurrences of X with Y in the stack and the environment and let extend the environment with $[X \mapsto Y]$ (N.B. that arrow denotes a binding of a type variable, not a procedure type). We have found an expansion of X, so we want to immediately expand it everywhere we encountered it and save the expansion for future use in the environment.
- If $Y \in tvar$, replace all occurrences of Y with X in the stack and the environment and let extend the environment with $[Y \mapsto X]$.
- If X and Y are both in $tproc$, such that $X = X_a \rightarrow X_r$ and $Y = Y_a \rightarrow Y_r$, push $X_a = Y_a$ and $X_r = Y_r$ onto the stack. These are both procedure type expressions, so their argument and return types must match each other. This is the type inference system's analog of `APPLY`.
- In any other case, the constraint $X = Y$ is unsatisfiable, indicating an error in the target program. Report this error and abort unification.

Constraint solving is a general problem in computer science. See the union-find algorithm for a generalized discussion of the problem and popular approach.

9.6 Subtypes

There are many ways of defining types; therefore there are many ways of defining a subtypes t' of t, which is written $t' <: t$ or $t' \subseteq t$. Here are a few:

- Informally, $t' <: t$ if you can use an instance of t' anywhere that you can use an instance of t
- Considering types as sets of values, the subset notation holds literally: t' is a subtype of t if it is also a subset of t
- By the Liskov substitution principle, if $q(x)$ is a property provable about objects x of type t, then $q(y)$ should be true for objects y of type t', where $t' <: t$ [Lis87].
- If t is defined by a predicate function $q(x)$ that returns true for $x \in t$, then $q(y)$ must also return true for every $y \in t'$.
- $t' <: t$ if t' is a sort of t

The subtype rule is:

$$\text{T-sub: } \frac{\Gamma \vdash exp : t' \quad t' <: t}{\Gamma \vdash exp : t} \tag{79}$$

9.6.1 Procedure Subtypes

Because $((\lambda\ ()\ exp_1)) \Rightarrow exp_1$, it must be the case that both sides of the reduction arrow have the same type. Let this type be r. By rule T-sub, we can substitute any expression exp_2 for exp_1 if $exp_2 : r'$ and $r' <: r$. Therefore a procedure of type $(a \rightarrow r')$ must be a subtype of a procedure of type $(a \rightarrow r)$. This is not surprising. We've just said that procedure return types are **covariant**: if a subtype relation holds on two procedure types, the same relationship must also hold on their return types. Procedure subtype argument types have a **contravariant** relationship, however: the procedure subtype must accept at least all values that the procedure base type accepted.

The procedure subtype rule is:

$$\text{T-proc-sub: } \frac{\Gamma \vdash exp : (a \rightarrow r') \quad a' <: a \quad r' <: r}{\Gamma \vdash exp : (a' \rightarrow r)} \tag{80}$$

9.6.2 Polymorphic Subtypes

In C++, Java, and most OOP languages, pointers are covariant.

In Java, arrays are covariant. This is a design flaw because it makes type checking of array access statically undecidable. Java must therefore check every array assignment at runtime, which makes it both inefficient and less checkable than languages that have no subtype relationship between arrays of subtypes.

Polymorphic types don't have a direct mathematical relationship in general.

10 Memory Management

Early languages had no memory management, e.g., FORTRAN required all arrays to have static size and prohibited recursive procedure calls. Later languages (notably first Algol, and soon thereafter, LISP) split memory into a dynamically controlled **stack** and **heap**. The stack grows and shrinks (...like a "stack" data structure) primarily in response to procedure calls. Values on the stack are no longer available after that **stack frame** is popped, e.g., when the procedure that created the frame returns. The heap is a giant block of random access memory (...unfortunately, *un*like a "heap" data structure). Values on the heap are available until deallocated.

With the introduction of the heap, a language needs a process for managing heap memory to ensure that programs don't ever allocate the same block of memory twice, and reclaim memory once it is no longer in use. Let a **block** of memory be a contiguous heap region, which can be described by a starting address and a size (e.g., in bytes). Let the **free list** and **allocated list** are each lists of block descriptors that have been, respectively, not allocated (i.e., are free) or allocated. Assume an implementation of these as doubly-linked lists. A **manual memory management** scheme provides the programmer with explicit procedures for allocating and deallocating heap memory. These might look like:

```
def alloc(numBytes):
    global freeList, allocList
    cur = freeList.firstNode()
    while cur != None:
        if cur.size greater than or equal to numBytes:
          block = BlockDescriptor(cur.start, numBytes)
          if (cur.size greater than numBytes):
              # Shrink the remaining block
              cur.size -= numBytes
              cur.start += numBytes
          else:
                # We used up the whole block
                freeList.remove(cur)

            allocList.insert(block)
            # return the block
            return block

    # We're␣out␣of␣memory!
␣␣␣␣return␣None

def free(block):
    global freeList, allocList
    allocList.remove(block)
    freeList.insert(block)
```

Many languages don't actually give you a block descriptor back from alloc; they return only the start address. In order to recover the block size, they often stash it (by convention) in the bytes right *before* that start address.

In C, `malloc` and `free` implement manual memory management. e.g.,

```
MyObj* ptr = (MyObj*)malloc(sizeof(MyObj));
...
free(ptr);
```

You can see why the implementation of memory allocation might be somewhat slow–the functions have to walk a linked list that initially contains one element, but increasingly becomes filled with lots of small block descriptors, and the most straightforward algorithm will not guarantee that sequentially allocated blocks are near enough each other in memory for hardware caches to function effectively.

Some problems with manual memory management:

1. Object initialization: type safety cannot be maintained if the program is free to treat objects and blocks of memory as interchangable.
2. Dangling pointers: If you hang onto an address to a block of freed memory and later try to use it, the result is undefined! That memory might have been allocated to some other object in the mean time.
3. Memory leaks: If you forget to free blocks of memory, eventually the program will run out of memory and be unable to continue (in practice, it will start swapping to disk memory and become unusuably slow before this occurs).
4. Memory management is boilerplate that clutters the program, without adding algorithmic insight.

In C++, `new` and `delete` allow for proper initialization of heap allocated objects. `new` allocates memory for an object and then invokes its constructor. Delete invokes the destructor and then frees memory. The destructor typically invokes `delete` recursively on all objects referenced by the one undergoing destruction. As a result, most C++ programmers will try to allocate objects on the stack whenever possible and create chains of destructors that automate much of the heap memory management. This addresses the first problem. But can we do better, building this type-safety automation right into the language and addressing the other drawbacks of memory management?

The allocation part is easy–values have to be explicitly created, so their allocation is necessarily part of the program. But when is the right time to free a block of memory? When the values in it will never be used again by the program. Unfortunately,

the function that determines whether a value will be used in the future is noncomputable

...it is as hard as evaluating the rest of the program and equivalent to the halting function. So we can at best conservatively approximate this function. One good approximation is that if no other variable is pointing at a value in the heap, then we know it can never be used again and can be freed. Algorithms for implementing this strategy are called **garbage collection** algorithms.

Reference counting is one garbage collection algorithm. In it, each object maintains a count of the number of incoming references (pointers). When this hits zero, the object automatically frees its own memory. This is used by many implementations of the Python language, by operating systems for managing shared libraries, and can be implemented in C++ using operator overloading to implement faux-pointers. Reference counting is very efficient and predictable because the program's action of setting a pointer to nulll triggers collection of exactly the objects that action allows to be freed. Reference counting has a key weakness: cycles of pointers ensure that even a completely disconnected component appears to always be in use. This is like two people each floating in space by holding the other one up...really the entire structure should crash (in our case, be deallocated).

One algorithm that can handle cycles is called **mark-sweep**:

```
def alloc2(size):
    x = alloc(size)
    if x == None:
        # Out of memory!  Time to collect.
        gc()
        x = alloc(size)
        if x == None:
            # After collecting, still not enough memory.
            growHeap()
            x = alloc(size)
    return x

def free2(block)
    # Do nothing!
    None

def gc():
    for x in programStack:
        mark(x)
    sweep()

def mark(block):
    if not block.marked:
        block.marked = True
        for x in block.pointers:
            mark(x)

def sweep():
    for x in allocList:
        if not x.marked:
            allocList.remove(x)
            freeList.add(x)
        else:
            x.marked = False
```

This is much more accurate than reference counting for identifying dead objects, but some problems remain. The first is that it has made the timing of garbage collection unpredictable. Some calls to `alloc` may take a very long time to return as they sweep the allocated part of the heap. The resulting pauses at runtime make this kind of garbage collection unacceptable for many realtime systems. In addition:

- Free memory becomes fragmented into little blocks over time.
- Memory becomes incoherent–structures allocated closely in time may be far apart in space, leading to poor cache behavior.
- We need run-time type information to identify pointers within blocks.
- How big should the heap initially be? If it is all of memory, that isn't fair to other programs in a multi-tasking OS. If it is too small, then we'll garbage collect too frequently in the beginning.

There are other algorithms that address some of these problems, like the **stop-and-copy** collector algorithm, which compacts all marked memory by moving objects around during collection, and **generational** collector algorithms that predict the lifetime of objects to minimize the size of the traced heap.

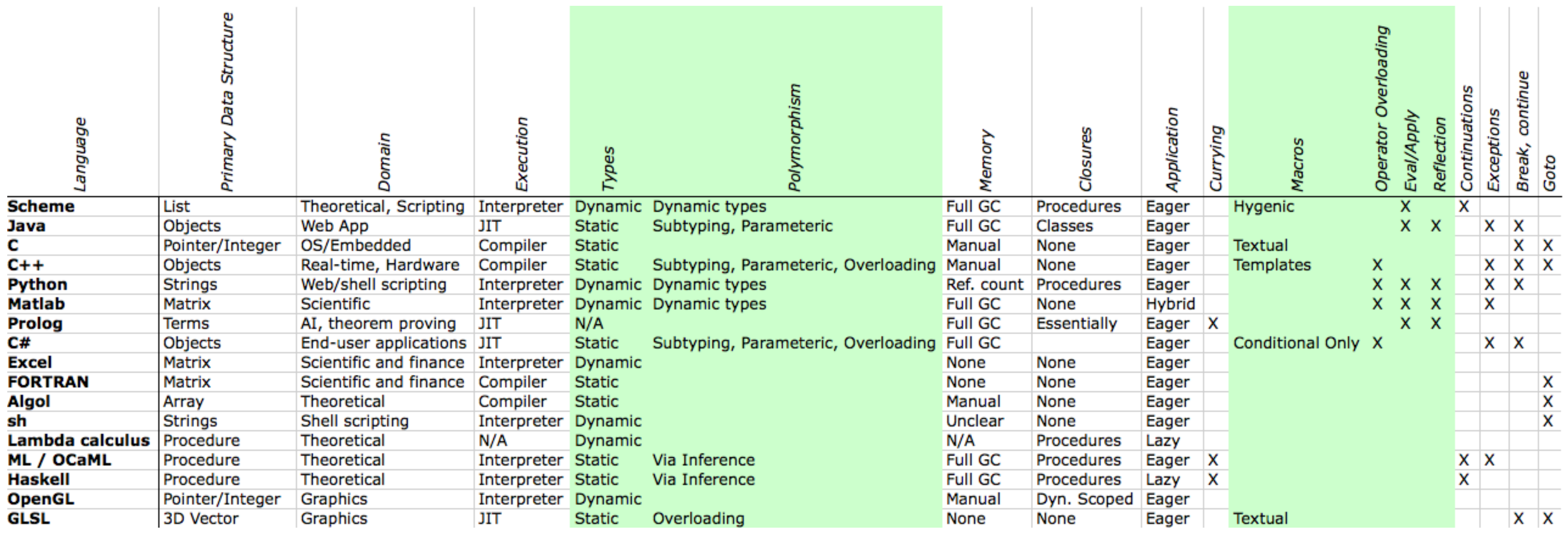
One interesting side note is **conservative collection**, which allows gc to operate on an implementation that does not provide run-time types. The way that it works is to treat *every* four bytes of allocated memory as if it were a legal pointer. Assuming block sizes are stashed right in front of the first byte of the block and that we have an allocated list, we can quickly reject many false pointers and for the rest can trace them through the system with mark-sweep. This is even more conservative than regular mark-sweep because sometimes four bytes will happen to match a valid block address when they are not in fact a pointer. Recall that mark-sweep could not determine whether a block will actually be used in the future, only whether it is referenced. So, the conservative collector is more conservative than regular mark-sweep, but neither is perfect. One popular implementation is the Boehm-Demers-Weiser collector http://www.hpl.hp.com/personal/Hans_Boehm/gc/, which is used to add garbage collection to C/C++.

11 Some Popular Languages

Every language is a point in a large design space of possible legal semantics (and syntax). It is interesting to compare some of the languages that have been heavily used in the past because they likely represent local maxima of practicality in that space. Furthermore, if we consider general use, both in production systems and for theoretical analysis, to be a large testing environment, then we can have the hope that the environment exerts a kind of natural selection on languages and that over time they are hopefully becoming better at trading expressivity and static checkability.

The table on the right summarizes the design choices of some popular language. Every language has many variants and implementations. Each evolves over time, and furthermore has particular subtleties that make such a table-summary fraught, at best. So please take this table as the broad overview that it is intended. The lack of version numbers is specifically intended to point up the "gist of the language" aspect, rather than detailing a particular version.

Often, no single language is appropriate for a project. For example, a computer graphics application like a video game might be built using Python and Make as scripting tools for the build system, Java or C# for building the 3D development tools, OpenGL and GLSL for graphics, Lua for AI scripting, and C++ as the glue code capable of interfacing with all of these and providing both high performance and syntax appropriate for expressing 3D mathematics.



Language	*Primary Data Structure*	*Domain*	*Execution*	*Types*	*Polymorphism*	*Memory*	*Closures*	*Application*	*Currying*	*Macros*	*Operator Overloading*	*Eval/Apply*	*Reflection*	*Continuations*	*Exceptions*	*Break, continue*	*Goto*
Scheme	List	Theoretical, Scripting	Interpreter	Dynamic	Dynamic types	Full GC	Procedures	Eager		Hygenic		X		X			
Java	Objects	Web App	JIT	Static	Subtyping, Parameteric	Full GC	Classes	Eager				X	X		X	X	
C	Pointer/Integer	OS/Embedded	Compiler	Static		Manual	None	Eager		Textual						X	X
C++	Objects	Real-time, Hardware	Compiler	Static	Subtyping, Parameteric, Overloading	Manual	None	Eager		Templates	X				X	X	X
Python	Strings	Web/shell scripting	Interpreter	Dynamic	Dynamic types	Ref. count	Procedures	Eager			X	X	X		X	X	
Matlab	Matrix	Scientific	Interpreter	Dynamic	Dynamic types	Full GC	None	Hybrid			X	X	X		X		
Prolog	Terms	AI, theorem proving	JIT	N/A		Full GC	Essentially	Eager	X			X	X				
C#	Objects	End-user applications	JIT	Static	Subtyping, Parameteric, Overloading	Full GC		Eager		Conditional Only	X				X	X	
Excel	Matrix	Scientific and finance	Interpreter	Dynamic		None	None	Eager									
FORTRAN	Matrix	Scientific and finance	Compiler	Static		None	None	Eager									X
Algol	Array	Theoretical	Compiler	Static		Manual	None	Eager									X
sh	Strings	Shell scripting	Interpreter	Dynamic		Unclear	None	Eager									X
Lambda calculus	Procedure	Theoretical	N/A	Dynamic		N/A	Procedures	Lazy									
ML / OCaML	Procedure	Theoretical	Interpreter	Static	Via Inference	Full GC	Procedures	Eager	X					X	X		
Haskell	Procedure	Theoretical	Interpreter	Static	Via Inference	Full GC	Procedures	Lazy	X					X			
OpenGL	Pointer/Integer	Graphics	Interpreter	Dynamic		Manual	Dyn. Scoped	Eager									
GLSL	3D Vector	Graphics	JIT	Static	Overloading	None	None	Eager		Textual						X	X

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