Greedy Approximations : $k$-Center

Algorithm Design & Analysis

Fall 2018
Outline
**k-Center Problem**

**Input:** $S = \{s_1, \ldots, s_n\} \subset \mathbb{R}^2$ and an integer $k$

**Problem:** Compute set $C$ of $k$ points in $\mathbb{R}^2$ that minimizes $\max_{s \in S} \{\text{dist}(s, C)\}$

$\text{dist}(s, C) = \min_{c \in C} \{|s - c|\}$

- $C$ is an $r$-cover of $S$ if $\text{dist}(s, C) \leq r$ for all $s \in S$.
- Given $C$, $r = \max_{s \in S} \{\text{dist}(s, C)\}$ is called the covering radius of $C$.
- Given $k$, we want a $C$ of size $k$ with smallest possible covering radius.

Infinite search space!
**k-Center Problem**

**Input:** \( S = \{s_1, \ldots, s_n\} \subseteq \mathbb{R}^2 \) and an integer \( k \)

**Problem:** Compute set \( C \) of \( k \) points in \( \mathbb{R}^2 \) that minimizes \( \max_{s \in S} \{dist(s, C)\} \)

- \( dist(s, C) = \min_{c \in C} \{|s - c|\} \)
**k-Center Problem**

**Input:** $S = \{s_1, \ldots, s_n\} \subset \mathbb{R}^2$ and an integer $k$

**Problem:** Compute set $C$ of $k$ points in $\mathbb{R}^2$ that minimizes $\max_{s \in S} \{\text{dist}(s, C)\}$

- $\text{dist}(s, C) = \min_{c \in C}\{|s - c|\}$
- $C$ is an $r$-cover of $S$ if $\text{dist}(s, C) \leq r$ for all $s \in S$. 

Given $C$, $r = \max_{s \in S}\{\text{dist}(s, C)\}$ is called the covering radius of $C$.

Given $k$, we want a $C$ of size $k$ with smallest possible covering radius.

Infinite search space!
**k-Center Problem**

**Input:** $S = \{s_1, \ldots, s_n\} \subset \mathbb{R}^2$ and an integer $k$

**Problem:** Compute set $C$ of $k$ points in $\mathbb{R}^2$ that minimizes $\max_{s \in S}\{\text{dist}(s, C)\}$

- $\text{dist}(s, C) = \min_{c \in C}\{|s - c|\}$
- $C$ is an $r$-cover of $S$ if $\text{dist}(s, C) \leq r$ for all $s \in S$.
- Given $C$, $r = \max_{s \in S}\{\text{dist}(s, C)\}$ is called the covering radius of $C$
**k-Center Problem**

**Input:** $S = \{s_1, \ldots, s_n\} \subset \mathbb{R}^2$ and an integer $k$

**Problem:** Compute set $C$ of $k$ points in $\mathbb{R}^2$ that minimizes $\max_{s \in S}\{\text{dist}(s, C)\}$

- $\text{dist}(s, C) = \min_{c \in C}\{|s - c|\}$
- $C$ is an $r$-cover of $S$ if $\text{dist}(s, C) \leq r$ for all $s \in S$.
- Given $C$, $r = \max_{s \in S}\{\text{dist}(s, C)\}$ is called the covering radius of $C$.
- Given $k$, we want a $C$ of size $k$ with smallest possible covering radius.
**k-Center Problem**

**Input:** $S = \{s_1, \ldots, s_n\} \subset \mathbb{R}^2$ and an integer $k$

**Problem:** Compute set $C$ of $k$ points in $\mathbb{R}^2$ that minimizes $\max_{s \in S}\{\text{dist}(s, C)\}$

- $\text{dist}(s, C) = \min_{c \in C}\{|s - c|\}$
- $C$ is an $r$-cover of $S$ if $\text{dist}(s, C) \leq r$ for all $s \in S$.
- Given $C$, $r = \max_{s \in S}\{\text{dist}(s, C)\}$ is called the *covering radius* of $C$
- Given $k$, we want a $C$ of size $k$ with smallest possible covering radius
- Infinite search space!
Bootstrapping to a Good Algorithm

Use the covering radius \( r^* \) and a cover \( C^* \) of size \( k \) with \( r(C^*) = r^* \) to find a good approximation that uses only sites as centers.

Algorithm 1

```plaintext
procedure GreedyKCenter1.0 (S, k, C*, r*) // C* covers S with optimal radius r*
C ← ∅
for all c ∈ C* do
    if c is within r* of some s ∈ S then
        Select some s within r* of c
        Add s to C; delete s from S
    Delete all s′ within 2r* of s from S
return C
end procedure
```
Bootstrapping to a Good Algorithm

Use the covering radius $r^*$ and a cover $C^*$ of size $k$ with $r(C^*) = r^*$ to find a good approximation that uses only sites as centers.

**Algorithm 2** Greedy $k$-Center 1.0

```plaintext
procedure GREEDYKCENTER1.0($S$, $k$, $C^*$, $r^*$) // $C^*$ covers $S$ with optimal radius $r^*$
    $C ← \emptyset$
    for all $c ∈ C^*$ do
        if $c$ is within $r^*$ of some $s ∈ S$ then
            Select some $s$ within $r^*$ of $c$
            Add $s$ to $C$; delete $s$ from $S$
            Delete all $s'$ within $2r^*$ of $s$ from $S$
        return $C$
    end procedure
```
Better Bootstrapping: Dropping \( C^* \)

The selected sites form a cover of \( S \) of radius \( 2r^* \)
**Better Bootstrapping : Dropping C**

The selected sites form a cover of $S$ of radius $2r^*$.

In fact, we don’t need $C$!

---

**Algorithm 4**

Greedy $k$-Center

1.5

```plaintext
procedure GreedyKCenter1.5(S, k, r*)
    // The optimal radius for $k$ centers is $r$
    C ← ∅
    while $S$ ≠ ∅
        Select some $s$ ∈ $S$
        Add $s$ to $C$; delete $s$ from $S$
        Delete all $s'$ within $2r^*$ of $s$ from $S$
    return $C$
end procedure
```

Knowing only $r^*$ can yield a 2-approximation.
**Better Bootstrapping : Dropping C***

The selected sites form a cover of $S$ of radius $2r^*$

In fact, we don’t need $C$!

Just pick next $s_i$ to be more than distance $2r^*$ from any site already in $C$. 

---

**Algorithm 5**

*Greedy k-Center*

1.5

```
procedure GreedyKCenter1.5(S, k, r*) // The optimal radius for k centers is r*
    C ← ∅
    while S ̸= ∅
        Select some $s$ ∈ S
        Add $s$ to $C$; delete $s$ from $S$
        Delete all $s'$ within $2r^*$ of $s$ from $S$
    return $C$
end procedure
```

Knowing only $r^*$ can yield a 2-approximation.
Better Bootstrapping: Dropping $C^*$

The selected sites form a cover of $S$ of radius $2r^*$

In fact, we don’t need $C$!

Just pick next $s_i$ to be more than distance $2r^*$ from any site already in $C$.

---

**Algorithm 6** Greedy $k$-Center 1.5

```plaintext
procedure GREEDYKCENTER1.5($S, k, r^*$) // The optimal radius for $k$ centers is $r^$

$C ← ∅$

while $S ≠ ∅$ do

Select some $s ∈ S$

Add $s$ to $C$; delete $s$ from $S$

Delete all $s'$ within $2r^*$ of $s$ from $S$

return $C$

end procedure
```
Better Bootstrapping: Dropping $C^*$

The selected sites form a cover of $S$ of radius $2r^*$

In fact, we don’t need $C$!

Just pick next $s_i$ to be more than distance $2r^*$ from any site already in $C$.

Algorithm 7 Greedy $k$-Center 1.5

```plaintext
procedure GREEDYKCENTER1.5($S, k, r^*$) // The optimal radius for $k$ centers is $r^*$
    $C \leftarrow \emptyset$
    while $S \neq \emptyset$ do
        Select some $s \in S$
        Add $s$ to $C$; delete $s$ from $S$
        Delete all $s'$ within $2r^*$ of $s$ from $S$
    return $C$
end procedure
```

Knowing only $r^*$ can yield a 2-approximation
Even Better Bootstrapping : Dropping $r^*$

Idea: Replace "Select some $s \in S" with "Select $s \in S$ furthest from $C" (initialize $C$ to any $s \in S$)
Even Better Bootstrapping : Dropping \(r^*\)

Idea: Replace "Select some \(s \in S\)" with "Select \(s \in S\) furthest from \(C\)" (initialize \(C\) to \(any s \in S\))

Algorithm 9 Greedy \(k\)-Center 2.0

\[
\begin{align*}
\text{procedure} & \quad \text{GREEDYKCENTER2.0}(S, k) \\
\text{if} & \quad k \geq |S| \text{ then} \\
& \quad \text{return } S \\
C & \leftarrow \text{some element } s \in S; \text{ delete } s \text{ from } S \\
\text{while} & \quad |C| < k \text{ do} \\
& \quad \text{Select some } s \in S \text{ of maximum distance from } C \\
& \quad \text{Add } s \text{ to } C; \text{ delete } s \text{ from } S \\
& \quad \text{return } C \quad \text{// Claim: } C \text{ is a } 2r^* \text{ cover of } S
\end{align*}
\]

end procedure
Why 2.0 Works

Theorem

If there is a set $C'$ of $k$ centers with covering radius $r$, then GreedyKCenter2.0 yields a covering $C$ of $S$ of size $k$ with $r(C) \leq 2r$. Otherwise there is no set $C'$ of $k$ centers with $r(C') = r$. 
Why 2.0 Works

Theorem
If there is a set \( C' \) of \( k \) centers with covering radius \( r \), then GreedyKCenter2.0 yields a covering \( C \) of \( S \) of size \( k \) with \( r(C) \leq 2r \). Otherwise there is no set \( C' \) of \( k \) centers with \( r(C') = r \).

Proof: It suffices to establish the following (obvious) property
Why 2.0 Works

**Theorem**
If there is a set $C'$ of $k$ centers with covering radius $r$, then GreedyKCenter2.0 yields a covering $C$ of $S$ of size $k$ with $r(C) \leq 2r$. Otherwise there is no set $C'$ of $k$ centers with $r(C') = r$.

**Proof:** It suffices to establish the following (obvious) property

**Property**
If there is a set $C'$ of $k$ centers with $r(C') = r$, then at the end of any iteration of the while loop, either $C$ is a cover of $S$ of radius $2r$ or there is an $s \in S$ of distance greater than $2r$ from $C$. 
Why 2.0 Works

Theorem
If there is a set $C'$ of $k$ centers with covering radius $r$, then GreedyKCenter2.0 yields a covering $C$ of $S$ of size $k$ with $r(C) \leq 2r$. Otherwise there is no set $C'$ of $k$ centers with $r(C') = r$.

Proof: It suffices to establish the following (obvious) property

Property
If there is a set $C'$ of $k$ centers with $r(C') = r$, then at the end of any iteration of the while loop, either $C$ is a cover of $S$ of radius $2r$ or there is an $s \in S$ of distance greater than $2r$ from $C$.

Thus, after $k$ iterations, either $C$ is a cover of radius $2r$, or $C \cup \{s\}$ is a set of $k + 1$ elements, all of which are pairwise more than distance $2r$ apart, contradicting existence of $C'$
Why 2.0 Works

Theorem
If there is a set $C'$ of $k$ centers with covering radius $r$, then GreedyKCenter2.0 yields a covering $C$ of $S$ of size $k$ with $r(C) \leq 2r$. Otherwise there is no set $C'$ of $k$ centers with $r(C') = r$.

Proof: It suffices to establish the following (obvious) property

Property
If there is a set $C'$ of $k$ centers with $r(C') = r$, then at the end of any iteration of the while loop, either $C$ is a cover of $S$ of radius $2r$ or there is an $s \in S$ of distance greater than $2r$ from $C$.
Thus, after $k$ iterations, either $C$ is a cover of radius $2r$, or $C \cup \{s\}$ is a set of $k + 1$ elements, all of which are pairwise more than distance $2r$ apart, contradicting existence of $C'$

Corollary
GreedyKCenter2.0 produces a set $C$ of $k$ centers with $r(C) \leq 2r^*$