NP-Completeness

Algorithm Design & Analysis

Spring 2019
Outline
Recap

• $X \leq_p Y$ if problem $X$ can be solved in polynomial time by some algorithm that is allowed to solve instances of problem $Y$ in constant time.

• Polynomial Equivalence: $X \equiv_p Y$ if $X \leq_p Y$ and $Y \leq_p X$.

• Transitivity: If $X \leq_p Y$ and $Y \leq_p Z$ then $X \leq_p Z$.

• Strategies for reductions
  - Direct equivalence ($\text{INDSET} \equiv_p \text{VERTEXCOVER}$)
  - Special Case: ($\text{VERTEXCOVER} \leq_p \text{SETCOVER}$)
  - Gadget Building: ($3\text{SAT} \leq_p \text{INDSET}$)

• Problem Characteristics
  - Decision Problems: Output is YES/NO
  - Certifiability: If answer is YES, there’s a “short proof”
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Recap: Decision Problems and Certifiers

- Algorithm \( A \) solves decision problem \( X \) in polynomial time if \( A(s) \) executes at most \( O(p(|s|)) \) operations, for some polynomial \( p() \)
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- A certifier $C(s, t)$ for decision problem $X$ is a polynomial-time certifier if...
Recap: Decision Problems and Certifiers

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  • $|t_s| \leq p(|s|)$ for some polynomial $p()$ ($t_s$ is not too big!)
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  - \( |t_s| \leq p(|s|) \) for some polynomial \( p() \) (\( t_s \) is not too big!)
  - \( C(s, t) \) runs in time \( q(|s|) \) for some polynomial \( q(x) \) (\( C() \) is efficient)
A Complexity Hierarchy

- \( P = \{ X : \text{There is a poly-time algorithm } A() \text{ that decides } X \} \)
- \( NP = \{ X : \text{There is a poly-time certifier } C(s, t) \text{ for } X \} \)

\[ \text{Claim: } P \subseteq NP \quad \text{C}(s, t) = A(s) \quad \text{just let } t = \epsilon \]

- \( \text{EXP} = \{ X : \text{Some exp-time algorithm } A() \text{ that decides } X \} \)

- \( \text{Claim: } NP \subseteq \text{EXP} \)
- \( P \subset \text{EXP} \) : Consequence of the Time Hierarchy Theorem

- \big\text{Big Question:}\big\  \text{Is } P = NP?\big\]
- \( \text{Consensus view is “no”} \)
- \( \text{Most fundamental problem in computer science} \)
- \( \text{Clay Foundation offers}$1,000,000 prize for the answer
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NP-Completeness Defined

Are there "hardest" problems in NP?
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Not obvious, but since $NP \subseteq EXP$, at least possible
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**Definition**

A decision problem \( X \) is **NP-Complete** if
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A decision problem $X$ is *NP-Complete* if

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**Definition**

A decision problem \( X \) is *NP-Complete* if

- \( X \in NP \)
- For every \( Y \in NP \), \( Y \leq_p X \)
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Are there "hardest" problems in NP?
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**Definition**
A decision problem $X$ is *NP-Complete* if

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Are there *any* such problems?
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**Definition**
A decision problem $X$ is *NP-Complete* if
- $X \in NP$
- For every $Y \in NP$, $Y \leq_p X$

Are there *any* such problems?

Surprisingly, *thousands* of problems have been shown to be NP-Complete
Why is Definition Important?
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**Theorem**

Let $Y$ be any NP-Complete problem. Then $Y \in P$ if and only if $P = NP$
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$P = NP \Rightarrow Y \in P$

- Clear: $Y \in NP$, so $Y \in P$
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**Theorem**

Let $Y$ be any NP-Complete problem. Then $Y \in P$ if and only if $P = NP$

**Proof.**

$P = NP \implies Y \in P$

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Let $Y$ be any NP-Complete problem. Then $Y \in P$ if and only if $P = NP$

Proof.

$P = NP \Rightarrow Y \in P$

- Clear: $Y \in NP$, so $Y \in P$

$Y \in P \Rightarrow P = NP$

- Let $X \in NP$. Then $X \leq_p Y$, since $Y$ is NP-Complete
Why is Definition Important?

**Theorem**

Let $Y$ be any NP-Complete problem. Then $Y \in P$ if and only if $P = NP$

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$Y \in P \Rightarrow P = NP$

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- But if $Y \in P$ then $X \in P$. 

□
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**Proof.**

$P = NP \Rightarrow Y \in P$

- Clear: $Y \in NP$, so $Y \in P$

$Y \in P \Rightarrow P = NP$

- Let $X \in NP$. Then $X \leq_p Y$, since $Y$ is NP-Complete
- But if $Y \in P$ then $X \in P$.
- Thus $NP \subseteq P$. But $P \subseteq NP$, so $P = NP$
Establishing NP-Completeness

There are two ways to show a problem \( Y \) is NP-Complete.
Establishing NP-Completeness

There are two ways to show a problem $Y$ is NP-Complete

*From Definition*

- Show that $Y \in \text{NP}$
- Show that for all $X \in \text{NP}$, $X \leq_p Y$

**Reduction**

- Show that $Y \in \text{NP}$
- Show that $Z \leq_p Y$ for some NP-Complete problem $Z$

So if $X \in \text{NP}$, $X \leq_p Z$ and $Z \leq_p Y$, so $X \leq_p Y$

Can’t use second method until we use first method!
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- So if \( X \in NP \), \( X \leq_p Z \) and \( Z \leq_p Y \), so \( X \leq_p Y \)

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There are two ways to show a problem \( Y \) is NP-Complete.

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- Show that for all \( X \in \text{NP}, X \leq_p Y \)

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- Show that \( Y \in \text{NP} \)
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- So if \( X \in \text{NP}, X \leq_p Z \) and \( Z \leq_p Y \), so \( X \leq_p Y \)

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Circuit Satisfiability: A First NP-Complete Problem

We will show the following

CIRCUITSAT is NP-Complete

CIRCUITSAT \leq_p ATMOST3SAT, so ATMOST3SAT is NP-Complete (ATMOST3SAT \in NP)

ATMOST3SAT \leq_p 3SAT, so 3SAT is NP-Complete (3SAT \in NP)

This will show that INDSET, VERTEXCOVER, SETCOVER are NP-Complete, since

They are all in NP, and

3SAT \leq_p INDSET \leq_p VERTEXCOVER \leq_p SETCOVER

From these, an avalanche of NP-Complete problems will follow
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Circuit Satisfiability : A First NP-Complete Problem

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Circuit Satisfiability: A First NP-Complete Problem

We will show the following

- **CIRCUITSAT** is NP-Complete
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- **ATMOST3SAT \( \leq_p \) 3SAT, so 3SAT is NP-Complete (3SAT \( \in \) NP)
- This will show that INDSET, VERTEXCOVER, SETCOVER are NP-Complete, since
  - They are all in NP, and
  - 3SAT \( \leq_p \) INDSET \( \leq_p \) VERTEXCOVER \( \leq_p \) SETCOVER
Circuit Satisfiability: A First NP-Complete Problem

We will show the following

- CIRCUITSAT is NP-Complete
- CIRCUITSAT $\leq_p$ ATMOST3SAT, so ATMOST3SAT is NP-Complete ($\text{ATMOST3SAT} \in \text{NP}$)
- ATMOST3SAT $\leq_p$ 3SAT, so 3SAT is NP-Complete ($\text{3SAT} \in \text{NP}$)
- This will show that INDSET, VERTEXCOVER, SETCOVER are NP-Complete, since
  - They are all in NP, and
  - 3SAT $\leq_p$ INDSET $\leq_p$ VERTEXCOVER $\leq_p$ SETCOVER
- From these, an avalanche of NP-Complete problems will follow
A Note on Alphabets and Strings

An alphabet $\Sigma$ is (just) a finite set. The inputs to decision problems are strings over some alphabet $\Sigma$. We write $\Sigma^*$ for the set of all finite strings over $\Sigma$ (including $\epsilon$).

A language $X$ over $\Sigma$ is just some $X \subseteq \Sigma^*$. So a decision problem $X$ is just a language over $\Sigma$. It doesn’t really matter which alphabet we choose.

If $|\Sigma| = k$, any $\sigma \in \Sigma$ can be encoded with $c = \lceil \log k \rceil$ bits.

So if $s \in \Sigma^*$ has length $n$, its bit-encoding has length $cn$ over \{0, 1\}.

So alphabet choice impacts efficiency by only a constant factor.
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The inputs to decision problems are strings over some alphabet $\Sigma$.

We write $\Sigma^*$ for the set of all finite strings over $\Sigma$ (including $\epsilon$).

A language $X$ over $\Sigma$ is just some $X \subseteq \Sigma^*$.

So a decision problem $X$ is just a language over $\Sigma$.

It doesn’t really matter which alphabet we choose.

If $|\Sigma| = k$, any $\sigma \in \Sigma$ can be encoded with $c = \lceil \log k \rceil$ bits.
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So alphabet choice impacts efficiency by only a constant factor.
Boolean Circuits: An Example

We will assume that $\Sigma = \{0, 1\}$ in what follows.
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Figure 8.4 A circuit with three inputs, two additional sources that have assigned truth values, and one output.
Definition

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A *boolean circuit* is a DAG in which

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- Other bits represent boolean operations ($\land, \lor, \neg$)

**Theorem** (We won’t prove this....)
Let $A$ be a poly-time algorithm that takes $n$ input bits and produces 1 output bit. Then there is a boolean circuit $C$ such that

- $C$ can be produced from $A$ in poly-time (and hence is of poly-size)
- $C$ produces a 1 if and only if $A$ does
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**Definition (CIRCUITSAT)**

Given a boolean circuit $C$ with $n$ input bits (some of which may be fixed), is there an assignment of values to the unfixed input bits such that $C$ returns 1 (true/yes)?
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Theorem

CIRCUITSAT is NP-complete
**Figure 8.4** A circuit with three inputs, two additional sources that have assigned truth values, and one output.
Proof.

Need to show that, for any \( X \in \text{NP} \), \( X \leq_p \text{CIRCUITSAT} \).

- Since \( X \in \text{NP} \), \( X \) has a poly-time certifier \( C(s, t) \).
- A string \( s \) is in \( X \) if and only if some \( t \) of length \( p(|s|) \) makes \( C(s, t) \) return "yes" (that is, 1).
- So view \( C(s, t) \) as an algorithm that takes at most \(|s| + p(|s|) \) bits and outputs 1 bit.
- \( C(s, t) \) can be converted into a boolean circuit \( C \) with \(|s| \) fixed bits; other \( p(|s|) \) bits represent \( t \).
- \( C \) is satisfiable if and only if there's some setting of \( t \) bits that makes \( C(s, t) \) true.
- Thus \( X \) has been poly-time reduced to an instance of \( \text{CIRCUITSAT} \).
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- Since $X \in NP$, $X$ has a poly-time certifier $C(s, t)$
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- Thus $X$ has been poly-time reduced to an instance of CIRCUITSAT
**From CIRCUITSAT to ATMOST3SAT**

**Definition**

Let $\Phi$ be a CNF expression with at most 3 literals per clause. ATMOST3SAT is the problem of deciding whether $\Phi$ is satisfiable.

**Theorem**

ATMOST3SAT is NP-complete.

**Proof.**

Note that ATMOST3SAT is in NP. We show that CIRCUITSAT $\leq_p$ ATMOST3SAT.

- Let $C$ be a boolean circuit. We'll build $\Phi_C$ such that $\Phi_C$ is satisfiable if and only if $C$ is satisfiable.
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Proof that $\text{CIRCUITSAT} \leq_p \text{ATMOST3SAT}$

Proof.

• For each fixed bit source $v$, create clause $(v)$ if value is 1 and $(\overline{v})$ otherwise.

• For output bit (sink) $v_{\text{final}}$, create clause $(v_{\text{final}})$ to force output bit to be 1.

• For each internal node $v$:
  • If $v$ is a $\neg$ gate from $u$, create clauses $(v \lor u) \land (\overline{v} \lor \overline{u})$.
  • If $v$ is a $\lor$ gate from $u$ and $w$, create clauses $(v \lor \overline{u}) \land (v \lor \overline{w}) \land (\overline{v} \lor u \lor w)$.
  • If $v$ is a $\land$ gate from $u$ and $w$, create clauses $(\overline{v} \lor u) \land (\overline{v} \lor w) \land (v \lor \overline{u} \lor \overline{w})$.

• If $\Phi$ is satisfiable, $(v_{\text{final}})$ has value 1, and all fixed source variables received their correct values, and all other source variables received values that make $C$ produce 1.

• If $C$ is satisfiable, $\Phi$ is satisfiable.
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