Flow-Augmenting Paths and Max-Flow Min-Cut Theorem

Algorithm Design & Analysis

Fall 2018
Outline
Max-Flow Min-Cut Theorem

Our goal for today

**Theorem**

Let $f$ be an $(s, t)$-flow on a network $G = (V, E, c)$ and let $(A, B)$ be an $(s, t)$-cut of $G$. Then $\nu(f) = \text{cap}(A, B)$ if and only if $f$ is a flow of maximum value and $(A, B)$ is a cut of minimum value.
Residual Graph of a Flow on $G$

Build associated graph from $G$ that includes *backward* edges.
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- $G_f = (V_f, E_f)$, the *residual flow graph* of $G = (V, E)$ and $f$, has $V_f = V$ and $E_f$ defined as follows:
  - For each $e \in E$ with $f(e) < c(e)$, $e$ is in $E_f$, and $e$ has capacity $c_f(e) = c(e) - f(e)$.
  - $e$ is a *forward* edge of $G_f$.
  - For each $e = (u, v) \in E$ with $f(e) > 0$, $e_R = (v, u)$ is in $E_f$ and $e_R$ has capacity $c_f(e_R) = f(e)$.
  - $e$ is a *backward* edge of $G_f$.

Let's do an example!
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- $G_f$ is a flow network built from $G$ and an $(s, t)$-flow on $G$.

- $G_f$ can be used to identify improvements or augmentations to $f$; that is, changes that will increase $\nu(f)$.
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- An edge in $G$ can become two edges in $G_f$.
- The ad-hoc pushing back of flow in $G$ preserves flow conservation at all nodes and corresponds to a directed $(s, t)$-path of positive flow in $G_f$.
- Such a path can be found by BFS from $s$ in $G_f$.
- The smallest capacity edge (bottleneck) on the path determines increase in flow in $G$. 

Flow-Augmenting Paths

Inspired by the demo....
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• Let $f$ be an $(s, t)$-flow in $G$, 

• Let $b$ be the smallest capacity among the edges of $P$, called the bottleneck of $P$. 

• So $b = c_f(\bar{e})$ for some $\bar{e} \in P$. 
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- let $P$ be an $(s, t)$-path in $G_f$ of positive capacity; that is, every edge of $e \in P$ has $c_f(e) > 0$. 

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Flows in Networks

Using \( P \) to Improve \( f \)

• If \( e \in E \) is a forward edge on \( P \), then add \( b \) to \( f(e) \) in \( G \).

• Note that \( 0 \leq f(e) + b \leq f(e) + c = f(e) + (c(e) - f(e)) = c(e)(\text{feasible}) \).

• If \( e \in E \) is a backward edge on \( P \), then \( (e_R = (v, u) \in P) \), subtract \( c \cdot f(e) \) from \( f(e) \) in \( G \).

• Note that \( c(e) \geq f(e) > f(e) - b \geq f(e) - c \cdot f(e) = f(e) - f(e) = 0 \)(feasible).
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- If $e \in E$ is a forward edge on $P$, then add $b$ to $f(e)$ in $G$. 

Note that:

- $0 \leq f(e) + b \leq f(e) + c f(e) = f(e) + (c(e) - f(e)) = c(e)$ (feasible)

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- At every vertex $v \neq \{s, t\}$ of $P$, considering $v$ in $G$, the flows of exactly 2 edges incident with $v$ have been changed. No other internal vertices of $G$ have had flow changed.
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  - $e_i$ is forward, $e_{i+1}$ is backward: So, $b$ additional units of flow enter $v_i$ from $e_i$ and $b$ fewer units of flow enter $v_i$ via $e_{i+1}^R$. 
  - Thus flow is conserved at every vertex of $G$.
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Cuts Defined by Flows

Summarizing

Theorem

If $G_f$ contains an $(s, t)$-path $P$ then $f$ is not a maximum flow. $P$ is called a flow-augmenting path. Put another way:

Theorem

If $f$ is a maximum flow then $G_f$ contains no flow-augmenting path.

A flow $f$ creates a natural partition $[A, B]$ in $G$ where $A$ consists of all vertices reachable from $s$ by a directed path in $G_f$. If $t \not\in A$ then $[A, B]$ is a cut, called the cut induced by $f$.

No flow-augmenting path in $G_f$ implies $t \not\in A$, so $[A, B]$ is a cut.
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Induced Cuts are Minimum Capacity

**Theorem**
Suppose flow \( f \) on \( G \) has no flow-augmenting path and let \([A, B]\) be the cut induced by \( f \). Then

\[
\begin{align*}
\text{for every } e = (u, v) \in [A, B], \quad f(e) &= c(e) \\
\text{for every } e \in [B, A], \quad f(e) &= 0
\end{align*}
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So by the Flow Value Lemma, \( \nu(f) = \cap(A, B) \): \( f \) is a maximum flow & \([A, B]\) is a minimum capacity cut.
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- for $e = (u, v) \in [A, B]$, if $f(e) < c(e)$, $u$ is reachable from $s$ in $G_f$ since $c_f(e) = c(e) - f(e) > 0$
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- for $e = (u, v) \in [A, B]$, if $f(e) < c(e)$, $u$ is reachable from $s$ in $G_f$ since $c_f(e) = c(e) - f(e) > 0$
- for $e \in [B, A]$, if $f(e) > 0$, $u$ is reachable from $s$ in $G_f$ since $e^R$ is in $G_f$ and $c_f(e^R) = f(e) > 0$
Pulling It All Together

To summarize

- If $G_f$ contains no flow-augmenting path, then there is a cut $[A, B]$ with $v(f) = \text{cap}(A, B)$.
- If there is a cut $[A, B]$ with $v(f) = \text{cap}(A, B)$ then $f$ is a maximum flow (and $[A, B]$ is a minimum capacity cut).
- If $f$ is a maximum flow, then $G_f$ has no flow-augmenting path.

Thus $f$ is a maximum flow if and only if $G_f$ has no flow-augmenting path if and only if $v(f) = \text{cap}(A, B)$ for some cut $[A, B]$. 
To summarize

- If $G_f$ contains no flow-augmenting path, then there is a cut $[A, B]$ with $\nu(f) = \text{cap}(A, B)$. 

- If there is a cut $[A, B]$ with $\nu(f) = \text{cap}(A, B)$, then $f$ is a maximum flow and $[A, B]$ is a minimum capacity cut.

- If $f$ is a maximum flow, then $G_f$ has no flow-augmenting path. Thus $f$ is a maximum flow if and only if $G_f$ has no flow-augmenting path if and only if $\nu(f) = \text{cap}(A, B)$ for some cut $[A, B]$. 
To summarize

- If $G_f$ contains no flow-augmenting path, then there is a cut $[A, B]$ with $v(f) = cap(A, B)$.
- If there is a cut $[A, B]$ with $v(f) = cap(A, B)$ then $f$ is a maximum flow (and $[A, B]$ is a minimum capacity cut).
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- If $G_f$ contains no flow-augmenting path, then there is a cut $[A, B]$ with $\nu(f) = \text{cap}(A, B)$.
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- If $G_f$ contains no flow-augmenting path, then there is a cut $[A, B]$ with $v(f) = \text{cap}(A, B)$.
- If there is a cut $[A, B]$ with $v(f) = \text{cap}(A, B)$ then $f$ is a maximum flow (and $[A, B]$ is a minimum capacity cut).
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Thus $f$ is a maximum flow if and only if $G_f$ has no flow-augmenting path if and only if $v(f) = \text{cap}[A, B]$ for some cut $[A, B]$. 