Dynamic Programming: Shortest Paths

Algorithm Design & Analysis

Fall 2018
Outline
The Problem

Allow negative edge costs in the shortest paths problem.

Assume no negative cost cycles.
The RoadMap

• Establish appropriate optimality recurrence
• Determine complexity
• Reduce space requirements
• Preserve ability to find path given reduced space
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The Bellman-Ford Algorithm

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Heads-Up:

• Path length refers to number of edges on path
• Path cost refers to sum of costs of edges on path

Note that
• $opt(i, v) = 0$ and $opt(0, v) = \infty$ if $v \neq t$.
• $opt(1, v) = c(v, t)$ if $(v, t) \in E$; $opt(1, v) = \infty$ otherwise.
• $opt(i, v) \leq opt(i - 1, v)$ for all $i > 0$. 
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Let $P$ be a minimum-cost path from $v$ to $t$ using at most $i$ edges.
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- If $P$ has length less than $i$, then $opt(i, v) = opt(i - 1, v)$.
- If $P$ has length $i$, $P$ consists of some edge $(v, u)$ and a path of length $i - 1$ from $u$ to $t$, so

\[ opt(i, v) = \min_{v, u \in E} \{ opt(i - 1, v), c(v, u) + opt(i - 1, u) \} \]
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Complexity Analysis

Observation: $opt(n - 1, s)$ is the cost of the optimal path from $s$ to $t$. 
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Space: $Opt[-, -]$ table takes $O(n^2)$ space.
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**Space:** $Opt[-,-]$ table takes $O(n^2)$ space.

**Time:** An entry of $Opt[-,-]$ might take $O(n)$ time to compute, for total of $O(n^3)$
**Better Time Complexity Analysis**

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- This sum is at most $m$, since each edge is used at most once.
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- There are $n$ rows to the table, so total time is $O(mn)$
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Let’s count table accesses in construction of $opt[\neg, \neg]$

- $opt[i, v]$ considers each neighbor of $v$, so it makes $outDeg(v)$ table accesses of $opt[\neg, \neg]$.
- Filling in $Opt[i, \neg]$ requires $\sum_{v \in V - \{t\}} outDeg(v)$ accesses.
- This sum is at most $m$, since each edge is used at most once.
- There are $n$ rows to the table, so total time is $O(mn)$
- Actual path can be extracted from table in $O(m)$ time or built into table.
Improving Memory Requirements

Observation: $opt[i, -]$ depends only on $opt[i - 1, -]$
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- Set $hold[v] \leftarrow \min_{(v, u) \in E} \{ opt[v], c(v, u) + opt[u] \}$.
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- Then set $opt[] \leftarrow hold[]$; repeat $n - 3$ more times

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**Observation:** \(\text{opt}[i, -]\) depends only on \(\text{opt}[i - 1, -]\)

- Use a 1-dim array \(\text{opt}[]\), initialized to \(\text{opt}[1, -]\), and a temporary array \(\text{hold}[]\).
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- Then set \(\text{opt}[] \leftarrow \text{hold}[]\); repeat \(n - 3\) more times
- This gives \(O(n)\) space complexity beyond the storing of the graph.
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How can we extract path now?
**Storing the Paths**

**Idea:** Add an array $next[v]$ holding vertex after $v$ on the current candidate shortest path from $v$ to $t$.

- $next[v]$ is initialized to `null` for all $v$
- If $opt[v]$ changes, update $next[v]$ to hold the next vertex on the new (shorter) path from $v$ to $t$.
- Let $T$ be the graph containing all edges $(v, next[v])$. $T$ is dynamically changing.
- Claim: $T$ is a tree throughout process.
- After $i^{th}$ iteration, $T$ contains shortest $v - t$ paths of length at most $i$. 
Proof that $T$ (ignoring edge directions) is a tree

First show that $|V(T)| - 1 = |E(T)|$. 

• Base Case: $T$ begins by containing \{t\} and no edges.

• Consider a point at which $\text{opt}[v]$ is being changed.

• Then $\text{opt}[v] > c(v, u) + \text{opt}[u]$ for some neighbor $u$ of $v$.

• So $u$ is in $T$ (since $\text{opt}[u] \neq \infty$).

• If $v$ is in $T$, then $\text{next}[v] = w \neq \null$ so $(v, w) \in T$ is replaced by $(v, u) \in T$.

• If $v$ is not in $T$, then we are adding a new vertex and a new edge to $T$.

Finally, observe that for every $v \in T$ there is a path from $v$ to $t$, so $T$ (undirected) is connected.
Proof that $T$ (ignoring edge directions) is a tree

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- **Consider a point at which** $\text{opt}[v]$ **is being changed**
- **Then** $\text{opt}[v] > c(v, u) + \text{opt}[u]$ **for some neighbor** $u$ **of** $v$
- **So** $u$ **is in** $T$ (**since** $\text{opt}[u] \neq \infty$)
- **If** $v$ **is in** $T$, **then** $\text{next}[v] = w \neq \text{null}$ **so** $(v, w) \in T$ **is replaced by** $(v, u) \in T$
- **If** $v$ **is not in** $T$, **then we are adding a new vertex and a new edge to** $T$
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- Assume updating $\text{opt}[v]$ creates a cycle in $T$
- Then the cycle looks like $v = v_0, v_1, \ldots, v_n = v$, where each $v_{i+1}(i < n) = \text{next}[v_i]$
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- By definition of $next[]$, $opt[v_0] > c(v_0, v_1) + opt[v_1]$
- Also $opt[v_i] = c(v_i, v_{i+1}) + opt[v_{i+1}]$, for all $i < n$
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\text{Thus } opt[v_0] > \left( \sum_{i=0}^{i=n-1} c(v_i, v_{i+1}) \right) + opt[v_n] \tag{1}
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\[
= \left( \sum_{i=0}^{i=n} c(v_i, v_{i+1}) \right) + opt[v_0], \text{where } v_{n+1} = v_0 \tag{2}
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- Assume updating $\text{opt}[v]$ creates a cycle in $T$
- Then the cycle looks like $v = v_0, v_1, \ldots, v_n = v$, where each $v_{i+1}(i < n) = \text{next}[v_i]$
- By definition of $\text{next}[]$, $\text{opt}[v_0] > c(v_0, v_1) + \text{opt}[v_1]$
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Thus \[
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- But this is a negative weight cycle! $\Rightarrow \Leftarrow$. 
Summary

- \( \text{opt} \) and \( \text{next} \) are of size \( n \) and each of \( n-2 \) updates takes \( O(m) \) time.
- Upon completion, \( \text{next}[v] \) contains first link in a cheapest path from \( v \) to \( t \).
- Total space required is \( O(m+n) \).
- Total time required is \( O(mn) \).
- Not quite as fast as Dijkstra, but more general.
Summary

- $\textit{opt}[-]$ and $\textit{next}[-]$ are of size $n$ and each of $n - 2$ updates takes $O(m)$ time.
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