Dynamic Programming I

Algorithm Design & Analysis

Fall 2018
Outline
Clarification: For solving recurrences, you may assume that \( n = b^k \) (\( b = 2 \), say) for simplicity. For proving algorithm correctness, you should account for all values of \( n \).
**Fibonacci Numbers**

\[
\text{fib}(n) = \begin{cases} 
1 & \text{if } n = 1, 2 \\
\text{fib}(n - 1) + \text{fib}(n - 2) & \text{otherwise}
\end{cases}
\]
Fibonacci Numbers

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Number of recursive calls made by \( \text{fib}(n) \)
Fibonacci Numbers

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1 & \text{if } n = 1, 2 \\
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Number of recursive calls made by \( fib(n) \)

\[ C(n) = \begin{cases} 
1 & \text{if } n = 1, 2 \\
1 + C(n - 1) + C(n - 2) & \text{otherwise}
\end{cases} \]
Fibonacci Numbers

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Thus \( C(n) \geq \text{fib}(n) \) for all \( n \geq 1 \) (simple induction proof)
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Well-known fact: \( \text{fib}(n) \geq \left( \frac{1 + \sqrt{5}}{2} \right)^{n-2} \geq 1.6^{n-2} \)
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So \(C(n) \geq 1.6^{n-2}\) for all \(n \geq 1\)

That is, \(C(n)\) grows exponentially!
Algorithm 1 Fibonacci

procedure FIB(n)
    if n ≤ 2 then return 1
    \( F_n = F_c = 1 \)
    for i ← 3 to n do
        \( t = F_n \)
        \( F_n = F_n + F_c \)
        \( F_c = t \)
    return \( F_n \)
end procedure
Algorithm 2 Fibonacci

```
procedure FIB(n)
  if n ≤ 2 then return 1
  \[ F_n = F_c = 1 \]
  for i ← 3 to n do
    \[ t = F_n \]
    \[ F_n = F_n + F_c \]
    \[ F_c = t \]
  return \( F_n \)
end procedure
```

But what if we are computing *many* Fibonacci numbers for repeated use in a program?
Algorithm 3 Fibonacci Table

procedure FIBTABLE(n)
    for $i \leftarrow 3$ to $i \leftarrow n$ do
        $F[i] = F[i - 1] + F[i - 2]$
    end procedure
Algorithm 4 Fibonacci Table

procedure FIBTABLE(n)
    for \( i \leftarrow 3 \) to \( i \leftarrow n \) do
        \[ F[i] = F[i - 1] + F[i - 2] \]
    end procedure

Another approach: Fill table opportunistically....
Algorithm 5 Fibonacci with Memoizing

procedure MEMO\textsc{Fib}(F, n) // Prior to first call, \(F[1..n]\) has been set to 0
    if \(F[n] > 0\) then
        return \(F[n]\)
    else if \(n = 1, 2\) then
        \(F[n] = 1\)
        return \(F[n]\)
    else
        \(F[n] = \text{memoFib}(F, n - 1) + \text{memoFib}(F, n - 2)\)
        return \(F[n]\)
end procedure
Recursive Fibonacci with Memoizing

Algorithm 6 Fibonacci with Memoizing

procedure MEMOFIB(F, n)// Prior to first call, F[1..n] has been set to 0
    if F[n] > 0 then
        return F[n]
    else if n = 1, 2 then
        F[n] = 1
        return F[n]
    else
        F[n] = memoFib(F, n − 1) + memoFib(F, n − 2)
        return F[n]
end procedure

Memoizing is very useful for making recursion more efficient!
Weighted Interval Scheduling

**The Input:** Given intervals \((s_1, t_1), \ldots, (s_n, t_n)\) where each \((s_i, t_i)\) has non-negative value (weight) \(v_i\).
Weighted Interval Scheduling

The Input: Given intervals \((s_1, t_1), \ldots, (s_n, t_n)\) where each \((s_i, t_i)\) has non-negative value (weight) \(v_i\).

The Output: A subset \(I \subseteq \{1, \ldots, n\}\), where the intervals \(\{(s_i, t_i) : i \in I\}\) are pairwise non-intersecting intervals that maximize \(\sum_{i \in I} v_i\).
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This notation is nicer than, say, \((s_{i_1}, t_{i_1}), \ldots, (s_{i_k}, t_{i_k})\) and \(\sum_{j=1}^{k} v_{i_j}\).
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Let’s simplify: Can we find the value of the best solution, not the actual set of intervals. That is, find the largest \(\sum_{i \in I} v_i\) where the intervals in \(I\) are compatible.
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Can we find a recurrence for \(\text{maxSched}(n)\)?
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- For $i \leq n$, let $maxSched(i)$ be value of the optimal schedule using only intervals in \{1, \ldots, i\}
- If $(s_n, t_n)$ isn’t used: $maxSched(n) = maxSched(n - 1)$
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But what if it is?

- If interval $(s_n, t_n)$ is used, then no interval $(s_j, t_j)$ with $j < n$ and $s_n \leq t_j \leq t_n$ is used (overlapping!)
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- So, for each $i > 1$ store the largest $j < i$ such that $t_j \leq s_i$ in a table: $p[i] = j$ (for predecessor)
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- So, for each $i > 1$ store the largest $j < i$ such that $t_j \leq s_i$ in a table: $p[i] = j$ (for predecessor)
- So only intervals $(s_j, t_j)$ with $j \leq p[n]$ can be used with $(s_n, t_n)$
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- So, for each $i > 1$ store the largest $j < i$ such that $t_j \leq s_i$ in a table: $p[i] = j$ (for predecessor)
- So only intervals $(s_j, t_j)$ with $j \leq p[n]$ can be used with $(s_n, t_n)$
- So if $(s_n, t_n)$ is used: $maxSched(n) = v_n + maxSched(p[n])$
Weighted Interval Scheduling

Algorithm 7
MaxSched with Memoizing

procedure maxSched (n)
// Prior to first call,
p[1..n] has been constructed
// And a table M[1..n] has been initialized to 0
if n = 0 then return 0
else if M[n] > 0 then return M[n]
else
    M[n] = max {maxSched(n - 1), v[n] + maxSched(p[n])}
return M[n]
end procedure
Weighted Interval Scheduling

\[\text{maxSched}(n) = \max\{\text{maxSched}(n - 1), v_n + \text{maxSched}(p(n))\}\]
Weighted Interval Scheduling

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Algorithm 9 MaxSched with Memoizing

\begin{procedure}
\textbf{MAXSCHED}(n)
\begin{scriptsize}
\hspace{1em} // Prior to first call, \( p[1..n] \) has been constructed
\hspace{1em} // And a table \( M[1..n] \) has been initialized to 0
\hspace{2em} if \( n = 0 \) then
\hspace{3em} return 0
\hspace{2em} else if \( M[n] > 0 \) then
\hspace{3em} return \( M[n] \)
\hspace{2em} else
\hspace{3em} \( M[n] = \max\{\text{maxSched}(n - 1), v_n + \text{maxSched}(p[n])\} \)
\hspace{3em} return \( M[n] \)
\end{scriptsize}
\end{procedure}
Algorithm 10 Iterated MaxSched

\begin{align*}
\textbf{procedure} & \quad \text{MAXSCHED}(n) \\
& \quad // \text{Prior to first call, } p[1..n] \text{ has been constructed} \\
& \quad M[0] = 0 \\
& \quad \textbf{for} \ i \leftarrow 1 \ \textbf{to} \ i \leftarrow n \ \textbf{do} \\
& \quad \quad M[i] = \max\{M[i - 1], v_i + M[p[i]]\} \\
\textbf{end procedure}
\end{align*}
Algorithm 11 Iterated MaxSched

**procedure** MAXSCHED(n)

// Prior to first call, p[1..n] has been constructed

\[ M[0] = 0 \]

for \( i \leftarrow 1 \) to \( i \leftarrow n \) do

\[ M[i] = \max\{ M[i - 1], v_i + M[p[i]] \} \]

end procedure

Notes

• If the intervals are sorted, \( p[1..n] \) can be built in \( O(n) \) time (convince yourselves!)

• Thus the algorithm takes \( O(n) \) space and \( O(n \log n) \) time.

• In fact, assuming we've sorted the (endpoints) of the intervals, it takes \( O(n) \) time!
Algorithm 12 Iterated MaxSched

procedure MAXSCHED(n)
    // Prior to first call, p[1..n] has been constructed
    M[0] = 0
    for i ← 1 to i ← n do
        M[i] = max\{ M[i - 1], v_i + M[p[i]] \}
    end procedure

Notes

• If the intervals are sorted, p[] can be built in $O(n)$ time (convince yourselves!)
Iterative Weighted Interval Scheduling

Algorithm 13 Iterated MaxSched

procedure MAXSCHED(n)
    // Prior to first call, p[1..n] has been constructed
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Iterative Weighted Interval Scheduling

How could we modify the method to also produce the optimal set of intervals?

Method 1:

- Build partial solutions.
  - Compute table $S[i]$, where $S[i]$ holds intervals in some optimal solution to $\text{maxSched}(i)$.
  - $S[i]$ can be built from $S[i-1]$ and $S[p[i]]$.

Changes run-time to $O(n^2)$.

Method 2:

- Reconstruct from $M[n]$.
  - If $v[n] + M[p[n]] \geq M[n-1]$ then include interval $n$ and recursively find rest of solution on intervals $\{1, \ldots, p[n]\}$.

Run-time remains $O(n)$ (after the initial sorting).
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**Method 2:** Reconstruct from $M[]$.

  - If $v_{n+M[p[n]]} \geq M[n-1]$ then include interval $n$ and recursively find rest of solution on intervals $\{1, \ldots, p[n]\}$.

  - Run-time remains $O(n)$ (after the initial sorting).
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Run-time remains $O(n)$ (after the initial sorting).
The Principle of Optimality

In the Weighted Interval Scheduling Problem we noted that either

• \( \text{maxSched}(i) = \text{maxSched}(i-1) \) (item \( i \) not used), or

• \( \text{maxSched}(i) = v_i + \text{maxSched}(p[i]) \) (item \( i \) was used)

This is an example of the Principle of Optimality

An optimal solution to the problem was built from optimal solutions to subproblems.

This is a common feature of many problems and is a powerful tool in the design of efficient algorithms!
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How To Build Predecessor Array

Assume list of combined start and finish times in increasing order. We will store the index of the largest end time \( t \) seen so far.

- Let \( p[1] = 0, t = 0 \).
- While there are unscanned items in the list
  - Consider the next item in the list, call it \( x \).
  - If \( x \) is a start time, then \( x = s_k \) for some \( k \); set \( p[k] = t \).
  - If \( x \) is an end time, then \( x = t_k \) for some \( k \). Update \( t \) to be \( k \).

This algorithm (clearly) takes \( O(n) \) time.
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