Implementing Kruskal’s Algorithm: Union-Find

Algorithm Design & Analysis

Spring 2018
Real Life Asymptotic Analysis

Geeks and repetitive tasks

- Geek: does it manually
- Non-geek: does it manually

- Gets annoyed
- Runs script

- Writes script to automate

- Makes fun of geek's complicated method

- Loses
- Wins

Time spent vs. task size
Moderate Greed: Prim’s Algorithm

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Prim maintains a subtree $T = (V', E')$ of $G$ and adds the cheapest cut edge of $E(V', V - V')$ (in $G$) to $T$. 

Algorithm 2

procedure Prim($G, c()$) // $G = (V, E)$ is connected
Select some $v ∈ V$; $V' ← \{v\}$; $T ← (V', \emptyset)$ // The eventual MCST
while $|E(T)| < |V| - 1$

Select cheapest edge $e ∈ E(V', V - V')$
Add $e$ to $T$ // This adds a new vertex to $T$
end procedure
Moderate Greed: Prim’s Algorithm

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**Algorithm 3 Prim’s Algorithm**

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procedure PRIM(G, c) // G = (V, E) is connected
  Select some $v \in V$; $V' \leftarrow \{v\}$; $T \leftarrow (V', \emptyset)$ // The eventual MCST
  while $|E(T)| < |V| - 1$ do
    Select cheapest edge $e \in E(V', V - V')$
    Add $e$ to $T$ // This adds a new vertex to $T$
  end procedure
```
Moderate Greed: Prim’s Algorithm

Algorithm 4 Prim’s Algorithm: More Detail

procedure PRIM\( (G, c()) \) // \( G = (V, E) \) is connected

Select some \( v \in V \); \( V' \leftarrow \{v\} \)
\( T \leftarrow (V', \emptyset) \) // The eventual MCST
\( C \leftarrow E(V', V - V') \)
// \( C \) will always contain \( E(V', V - V') \)

while \( |E(T)| < |V| - 1 \) do

Select cheapest edge \( e \in C \)
if \( e \in E(V', V - V') \) then
// Let \( e = \{u, v\} \), where \( u \in V' \) and \( v \in V - V' \)
Add \( v \) to \( T \); Add \( e \) to \( T \)
for neighbors \( w \) of \( v \) do
if \( w \in V - V' \) then add \( \{v, w\} \) to \( C \)
end procedure
The Implementation

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Theorem

Prim’s algorithm uses $O(|V| + |E|)$ space and runs in $O((|V| + |E|) \log |E|)$ time
The Implementation

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*Prim’s algorithm uses $O(|V| + |E|)$ space and runs in $O((|V| + |E|) \log |E|)$ time*

Proof: The loop executes at most $|E|$ times and each iteration performs a constant number of operations other than addition to (or removal from) $C$, a heap of size $O(|E|)$. 
The Implementation

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Prim’s algorithm uses \( O(|V| + |E|) \) space and runs in \( O((|V| + |E|) \log |E|) \) time

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But the total number of heap operations is \( O(|E|) \) and each takes time \( O(\log |E|) \).

Note: Since \( |E| \in O(|V|^2) \), \( \log |E| \in O(\log |V|) \)
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• $C$ is stored as a priority queue with edge weight as priority
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Note: Since $|E| \in O(|V|^2)$, $\log |E| \in O(\log |V|)$
So Prim runs in time $O((|V| + |E|) \log |V|)$ time.
Keep adding cheapest edge that doesn’t create a cycle.

Algorithm 5 Kruskal’s Algorithm

procedure KRUSKAL(G, c()) // G = (V, E) is connected
  T ← (V, ∅) // The eventual MCST
  F ← E
  while |E(T)| < |V| − 1 do
    Remove cheapest edge e ∈ F from F
    if T + {e} does not contain a cycle then
      Add e to T
  end procedure
To implement Kruskal’s Algorithm, we want efficient methods to

- Find the cheapest edge in \( F \)
- Determine whether \( T + \{ e \} \) contains a cycle
- Add an edge to \( T \)

To find the cheapest edge in \( F \), we can store \( F \) as a min-heap.

To create the heap, we store \( F \) as an array and heapify it.

How should we heapify it: bottom-up or top-down?
The Implementation

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How should we heapify it: bottom-up or top-down?
**Bottom-Up Heapify**

How to turn an array into a heap quickly

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**Algorithm 6** Bottom-up Heapify

```plaintext
procedure BUHEAPIFY(H[])
    // Ignore H[0]; n is largest value with H[n] not empty
    // H[n/2 + 1..] have no children with data
    for i ← n/2 down to i ← 1 do
        HeapifyDown(H,i)
end procedure
```

---

**Correctness:**
- Before the loop executes, $H[n/2 + 1..n]$ has the heap property.
- By induction, just before we call HeapifyDown($H,i$), $H[i+1..n]$ satisfies the heap property.
- So, after the call, $H[i..n]$ satisfies the heap property.
**Bottom-Up Heapify**

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**Algorithm 7 Bottom-up Heapify**

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• By induction, just before we call HeapifyDown(H, i), H[i + 1...n] satisfies the heap property.
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- So, after the call, $H[i..n]$ satisfies the heap property.
Time Complexity of Bottom-Up Heapify

A binary tree of depth $k$ has at most $m = 2^k + 1 - 1$ nodes:

$$m = k \sum_{i=0}^{d-1} \# \text{of elements at depth } i = k \sum_{i=0}^{d-1} 2^i = 2^k + 1 - 1$$

If the heap has depth $d$, it has $2^d \leq n \leq 2^d + 1 - 1$ nodes:

- There are $2^d - 1$ nodes of depth at most $d - 1$
- There are at most $2^d$ additional nodes at depth $d$, for a total of at most $2^d + 2^d - 1 = 2^d + 1 - 1$ nodes

A node at depth $k$ will be swapped at most $d - k$ times, so the total number of swaps is

$$k \sum_{k=0}^{d-1} \# \text{of nodes at depth } k \times \# \text{of swaps} \leq k \sum_{k=0}^{d-1} (d - k) 2^k \leq 2n - \lfloor \log n \rfloor - 2 \in O(n)$$
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$$\sum_{k=0}^{k=d} (\# \text{ of nodes at depth } k) \times (\# \text{ of swaps}) \leq \sum_{k=0}^{k=d} (d - k)2^k$$
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Union-Find: Cycle Checking and Tree Merging

Cycle Checking: Given the next edge $e \in F$, does $T + \{e\}$ contain a cycle?

Idea:
Union-Find: Cycle Checking and Tree Merging

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**Idea**:  
- Maintain a partition of $V$ based on components of $T$
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- Denote by $V_u$ the set in the partition containing vertex $u$. 

Recall: For $u, w \in V$, either $V_u = V_w$ or $V_u \cap V_w = \emptyset$ (Equivalence classes)

Adding an edge $e = \{u, w\}$ from $F$ to $T$ creates a cycle iff $V_u = V_w$.

So we need to be able to determine whether $V_u = V_w$.

And we need to be able to merge $V_u$ with $V_w$ to “add” $e$ to $T$. 
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A First Union-Find Structure

Union-Find Data Structure
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- Manages a dynamic partition of a set $S$

Kruskal's Algorithm can then use Find for cycle checking and Union to update the structure after adding an edge to $T$. 
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  $\text{MakeUnionFind}()$: Initialize the structure

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- Manages a dynamic partition of a set $S$
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  \[
  \text{MakeUnionFind}(): \text{Initialize the structure} \\
  \text{Find}(x): \text{Return name of set containing } x
  \]
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  \[ \text{MakeUnionFind}() : \text{Initialize the structure} \]

  \[ \text{Find}(x) : \text{Return name of set containing } x \]

  \[ \text{Union}(X, Y) : \text{Replace sets } X \text{ and } Y \text{ of partition with } Z = X \cup Y. \]
A First Union-Find Structure

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  $\text{MakeUnionFind}()$: Initialize the structure
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  $\text{Union}(X, Y)$: Replace sets $X$ and $Y$ of partition with $Z = X \cup Y$.

Kruskal’s Algorithm can then use Find for cycle checking and Union to update the structure after adding an edge to $T$. 
Extreme greed: Kruskal’s Algorithm

Keep adding cheapest edge that doesn’t create a cycle.

**Algorithm 11** Kruskal’s Algorithm

```plaintext
procedure KRUSKAL(G, c()) // G = (V, E) is connected
    T ← (V, ∅) // The eventual MCST
    F ← E
    MakeUnionFind(V)
    while |E(T)| < |V| − 1 do
        Remove cheapest edge e = \{u, v\} ∈ F from F
        uName = Find(u); vName = Find(v)
        if uName ≠ vName then
            Add e to T
            Union(uName, vName)
    end procedure
```
Let $S = \{1, \ldots, n\}$ be our set of items
Union-Find Implementation

Let $S = \{1, \ldots, n\}$ be our set of items

- $\text{MakeUnionFind}()$ creates one set for each vertex $v \in S$; the name of set is the name of the vertex.
Let $S = \{1, \ldots, n\}$ be our set of items

- $\textit{MakeUnionFind}()$ creates one set for each vertex $v \in S$; the name of set is the name of the vertex.
  - We can use an array $\textit{UFSets}[1 \ldots n]$ to hold the names: $\textit{UFSets}[v] = v$: $O(n)$ time

- $\textit{Find}(v)$ works by looking up the name of the set containing $v$ in the array $\textit{UFSets}[1 \ldots n]$:
  - $O(1)$ time

- $\textit{Union}(X, Y)$: $X \cup Y$ gets the name of whichever set $X$ or $Y$ is larger (ties are broken arbitrarily)
  - $\textit{Union}(X, Y)$ changes the names of each of the elements in the smaller set to the name of the larger set:
    - $O(n)$ time

- Doing this changes fewer names
- Keeping linked lists of the elements of each set makes it easy to find the elements whose names need changing

Let's try an example
Let $S = \{1, \ldots, n\}$ be our set of items

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Let’s try an example
A Surprising Fact

Lemma Any initial sequence of \( k \) Union's takes total time \( O(k \log k) \)

Proof

- The first time the algorithm refers to a vertex, pay the vertex $1.
- Every time the algorithm changes the set name of a vertex, pay the vertex $1.

Claim: After \( k \) Unions, total payout is at most \( O(k \log k) \).

Note that:
- At most \( 2k \) vertices are ever given any money, since a Union can give at most 2 vertices their first dollar.
- Each such vertex is now in a set of size greater than 1—but the union of all of those sets can have size no greater than \( 2k \).
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Our First Union-Find Theorem

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*Union-Find can be implemented so that MakeUnionFind takes \(O(n)\) time, Find takes \(O(1)\) time and any initial sequence of \(k\) Unions takes \(O(k \log k)\) time.*
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**Corollary**

*Kruskal’s Algorithm can be implemented to run in $O(m \log m)$ time. [Repeatedly deleting from the heap is the bottleneck.]*
An Improvement?

Can we rename fewer vertices during a Union?

- UFSets[\[v\]] needn't be \[w\].

- UFSets[\[] encodes a tree for each set \[X\] in our partition.

- The root of the tree for \[X\] is name of \[X\].

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- We'll only call \[\text{Union}(x, y)\] when \[x\] and \[y\] are vertices that name sets.

- \[\text{Union}(x, y)\] creates a set named by larger of the two sets.

- Thus, if the set named \[x\] is larger, \[\text{UFSets}[\[y\]] \leftarrow x\]. (Set \[y\] now points to set \[x\]).

- So, \[\text{Union}\] now takes \[O(1)\] time!

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Using path compression, any initial sequence of $m$ Union and Find operations on $n$ items after a MakeUnionFind can be carried out in $O(n + m \log^* n)$ time.
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But What is This $\log^* n$ Function?

Definition

For any base $b > 1$, $\log^* n$ is the number of times $\log_b$ must be repeatedly applied to $n$ before the result is less than 1. Precisely:

$$\log^*(n) = \begin{cases} 
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<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>4 = $2^2$</th>
<th>16 = $2^4$</th>
<th>65,536 = $2^{16}$</th>
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</tr>
</thead>
<tbody>
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<td>0</td>
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