Directed Graphs & Applications

Algorithm Design & Analysis

Fall 2018
Outline

Connectivity and Traversals in Directed Graphs

Applications

Deciding Strong Connectivity
DAGs and Topological Sorting
Reachability: An Equivalence Relation
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**Transitive** For all $x, y, z \in X$, $x \simeq y$ and $y \simeq z \implies x \simeq z$
Equivalence Relations ⇔ Equivalence Classes

An equivalence relation on a set $S$ gives rise to equivalence classes $S_x = \{ y : y \text{ is equivalent to } x \}$. These equivalence classes have the following properties:

- For every $x \in S$, $x \in S_x$.
- For every $x, y \in S$, either $S_x = S_y$ or $S_x \cap S_y = \emptyset$. That is, the equivalence classes partition $S$.

Alternate notation for $S_x$: $[x]$. For an undirected graph $G = (V, E)$, reachability is an equivalence relation on $V$—for each $v \in V$, $[v]$ is the set of vertices in the connected component of $G$ containing $v$. 
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Connectivity in Directed Graphs

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Two vertices $u, v$ in a directed graph $G$ are *mutually reachable* if there is a directed path from $u$ to $v$ and one from $v$ to $u$. 
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That is, $u, v$ are mutually reachable if $v$ is reachable from $u$ and $u$ is reachable from $v$. 
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A graph $G$ is *strongly connected* if every pair of vertices are mutually reachable.
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The Mutual Reachability relation decomposes $G$ into *strongly connected components*.
**Strong Components: An Example**

A graph and its strongly connected components
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**Analysis**

- **First step:** one call to BFS: $O(n + m)$ time.
- **Second step:** $n - 1$ calls to BFS: $O(n \times (n + m))$ time.

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Idea: Flip all the edges of $G$ and call BFS on $v$ again. Precisely

Let $G_{\text{rev}} = (V, E_{\text{rev}})$, where $(u, v) \in E_{\text{rev}}$ if $(v, u) \in E$.

Observe: There is a directed path from $v$ to $u$ in $G_{\text{rev}}$ iff there is a directed path from $u$ to $v$ in $G$.

So call BFS($G_{\text{rev}}, v$): Every vertex is reachable from $v$ (in $G_{\text{rev}}$) if and only if $v$ is reachable from every vertex (in $G$).

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- BFS($G, v$): $O(n+m)$ time
- Build $G_{\text{rev}}$: $O(n+m)$ time. [Do you believe this?]
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DAGs and Topological Sorting
**Application: Topological Sorting**

**Definition**

A directed graph is *acyclic* (or a *DAG*) if it contains no directed cycles.
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An ordering $v_1, v_2, \ldots, v_n$ of the vertices of a directed graph $G = (V, E)$ is a topological ordering if every edge $(v_i, v_j) \in E$ satisfies $i < j$. 
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Can we find one?
DAG and Topological Order: An Example
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Finding a Topological Order for a DAG

Claim
Every DAG G has a vertex with in-degree (out-degree) 0.
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Consider a simple path of maximum length (# of edges)
$P = u = v_0, v_1, \ldots, v_n = v$. 
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Idea
Build order by repeatedly removing a vertex of in-degree 0 from $G$. 
Topological Sorting Algorithm

Algorithm 1 Topological Sorting

procedure TS(G) ▷ G = (V, E) is a DAG
    T[1..n] ← 0; i ← 0
    while V is not empty do
        i ← i + 1
        Find a vertex v ∈ V with indeg(v) = 0
        T[i] ← v
        Delete v (and its edges) from G
    end while
end procedure

Prove correctness by induction on n: If G is a DAG, so is G − v.
Finding $v$ quickly

1. Compute in-degrees $ID[1..n]$ of all vertices using BFS of $G$:
   $O(n+m)$ time \[\text{Because BBFS can visit every edge!}\]

2. Scan $ID[]$ to produce a set $S$ of all vertices of in-degree 0:
   $O(n)$ time

3. Update $S$: When $v$ is deleted, decrement $ID[u]$ for each neighbor $u$; if $ID[u] = 0$, add $u$ to $S$:
   $O(\text{outdeg}(v))$ time

4. Total time for previous step over all vertices:
   $\sum_v \in V c \times \text{outdeg}(v) = c \sum_v \in V \text{outdeg}(v) = c \times m$:
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Result: Topological Sorting takes $O(n+m)$ time and space!
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