**Algorithms: Max Flow**

Consider the graph in Model 1 and the algorithm below. The algorithm claims to find in a graph $V, E$ the maximum flow $f(e) \forall e \in E$ through the graph. On line 3, it arbitrarily chooses an unsaturated path and adds flow to it. What is an ordering of choices that enables this algorithm to successfully find a maximum flow?

1. Initialize $f(e) \leftarrow 0$ for all $e \in E$
2. repeat
3. Find an unsaturated path $P$ from $s$ to $t$
4. $a \leftarrow$ minimum excess capacity $c(e) - f(e)$ among all edges $e \in P$
5. $f(e) \leftarrow f(e) + a$ for each edge $e \in P$
6. until no more unsaturated $s \leftarrow t$ paths

What is an ordering of choices that prevents this algorithm from finding a maximum flow?

**Learning objective:** Students will derive an efficient algorithm for finding the flow of maximum value in a graph.

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**Model 1: Graph**

![Graph Diagram]
Model 2: Flow Graphs
**Definition 1.** Given a network \( G = (V, E) \) and a flow \( f \) on \( G \), we define the residual graph \( G_f \) as \( G_f = (V, E_f) \), with

\[
E_f = \{ e \in E \mid c(e) - f(e) > 0 \} \cup \{ e^R \mid e \in E, f(e) > 0 \}
\]

(where \( e^R \) denotes the reverse of edge \( e \), i.e., if \( e = (u, v) \), then \( e^R = (v, u) \) and we define the capacities of edges in \( E_f \) by

- \( c_f(e) = c(e) - f(e) \), and
- \( c_f(e^R) = f(e) \).

3 Based on the given definition of a residual graph, draw a residual graph for each of the flow graphs depicted in Model 2.
4. Consider the algorithm below.

1. Initialize $f(e) \leftarrow 0$ for all $e \in E$
2. repeat
3. $\alpha \leftarrow \min \{c_f(e) \mid e \in P\}$
4. $f(e) \leftarrow f(e) + \alpha$ for each $e \in P$ such that $e \in E$
5. $f(e) \leftarrow f(e) - \alpha$ for each $e \in P$ such that $e^R \in E$
6. until no paths exist from $s$ to $t$ in $G_f$

This algorithm, the Ford-Fulkerson algorithm, uses the residual graph $G_f$ instead of the original graph to find augmenting paths. We execute lines 3, 4 and 5 because we have found an augmenting path in the residual graph.

Use the Ford-Fulkerson algorithm to find a flow on the graph in Model 3. Select paths in any order you like.

Model 3: Original Graph

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5. Is the flow you computed in the previous question the maximum possible flow? Why or why not?