CSCI 136
Data Structures & Advanced Programming

Lecture 31

Fall 2018

Instructors: Bills

Last Time

- Greedy Algorithms for Optimization
- Lab 10 : Exam Scheduling
- Adjacency List Implementation Details

Today's Outline

- GraphList Wrap-up
- An Important Algorithm: Minimum-cost spanning subgraph

GraphListVertex Iterators

```
// Iterator for incident edges
public Iterator<Edge<V,E>> adjacentEdges() {
        return adjacencies.iterator();
}

// Iterator for adjacent vertices
public Iterator<V> adjacentVertices() {
        return new GraphListAIterator<V,E>
            (adjacentEdges(), label());
}
```

GraphListAIterator creates an Iterator over *vertices* based on the Iterator over *edges* produced by `adjacentEdges()`

GraphListAIterator

GraphListAIterator uses two instance variables

```
    protected AbstractIterator<Edge<V,E>> edges;
    protected V vertex;

public GraphListAIterator(Iterator<Edge<V,E>> i, V v) {
        edges = (AbstractIterator<Edge<V,E>>)i;
        vertex = v;
}

public V next() {
        Edge<V,E> e = edges.next();
        if (vertex.equals(e.here()))
            return e.there();
        else { // could be an undirected edge!
            return e.here();
}
```

GraphListEIterator

GraphListEIterator uses one instance variable

```
protected AbstractIterator<Edge<V,E>> edges;
```

GraphListEIterator
- Takes the Map storing the vertices
- Uses it to build a linked list of all edges
- Gets an iterator for this linked list and stores it, using it in its own methods

GraphList

- To implement GraphList, we use the GraphListVertex (GLV) class
- GraphListVertex class
  - Maintain linked list of edges at each vertex
  - Instance vars: label, visited flag, linked list of edges
- GraphList abstract class
  - Instance vars:
    - Map<V,GraphListVertex<V,E>> dict; // label -> vertex
    - boolean directed; // is graph directed?
- How do we implement key GL methods?
  - GraphList(), add(), getEdge(), …

```
protected GraphList(boolean dir){
      dict = new Hashtable<V,GraphListVertex<V,E>>();
      directed = dir;
}

public void add(V label) {
      if (dict.containsKey(label)) return;
      GraphListVertex<V,E> v = new
            GraphListVertex<V,E>(label);
      dict.put(label,v);
}

public Edge<V,E> getEdge(V label1, V label2) {
      Edge<V,E> e = new Edge<V,E> (get(label1),
      get(label2), null, directed);
      return dict.get(label1).getEdge(e);
}
```

GraphListDirected

- GraphListDirected (GraphListUndirected) implements the methods requiring different treatment due to (un)directedness of edges
  - addEdge, remove, removeEdge, …

```java
// addEdge in GraphListDirected.java
// first vertex is source, second is destination
 public void addEdge(V vLabel1, V vLabel2, E label) {
    // first get the vertices
   GraphListVertex<V,E> v1 = dict.get(vLabel1);
   GraphListVertex<V,E> v2 = dict.get(vLabel2);
    // create the new edge
   Edge<V,E> e = new Edge<V,E>(v1.label(), v2.label(), label, true);
    // add edge only to source vertex linked list (aka adjacency list)
    v1.addEdge(e);
}
```

```java
public V remove(V label) {
   //Get vertex out of map/dictionary
   GraphListVertex<V,E> v = dict.get(label);

   //Iterate over all vertex labels (called the map "keyset")
   Iterator<V> vi = iterator();
   while (vi.hasNext()) {
        //Get next vertex label in iterator
        V v2 = vi.next();

        //Skip over the vertex label we're removing
        //(Nodes don't have edges to themselves...)
        if (!label.equals(v2)) {
            //Remove all edges to "label"
            //If edge does not exist, removeEdge returns null
            removeEdge(v2,label);
        }
    }
    //Remove vertex from map
    dict.remove(label);
    return v.label();
}
```

```java
public E removeEdge(V vLabel1, V vLabel2) {
    //Get vertices out of map
    GraphListVertex<V,E> v1 = dict.get(vLabel1);
    GraphListVertex<V,E> v2 = dict.get(vLabel2);

    //Create a “temporary” edge connecting two vertices
    Edge<V,E> e = new Edge<V,E>(v1.label(), v2.label(), null, true);

    //Remove edge from source vertex linked list
    e = v1.removeEdge(e);
    if (e == null) return null;
    else return e.label();
}
```

Efficiency Revisited

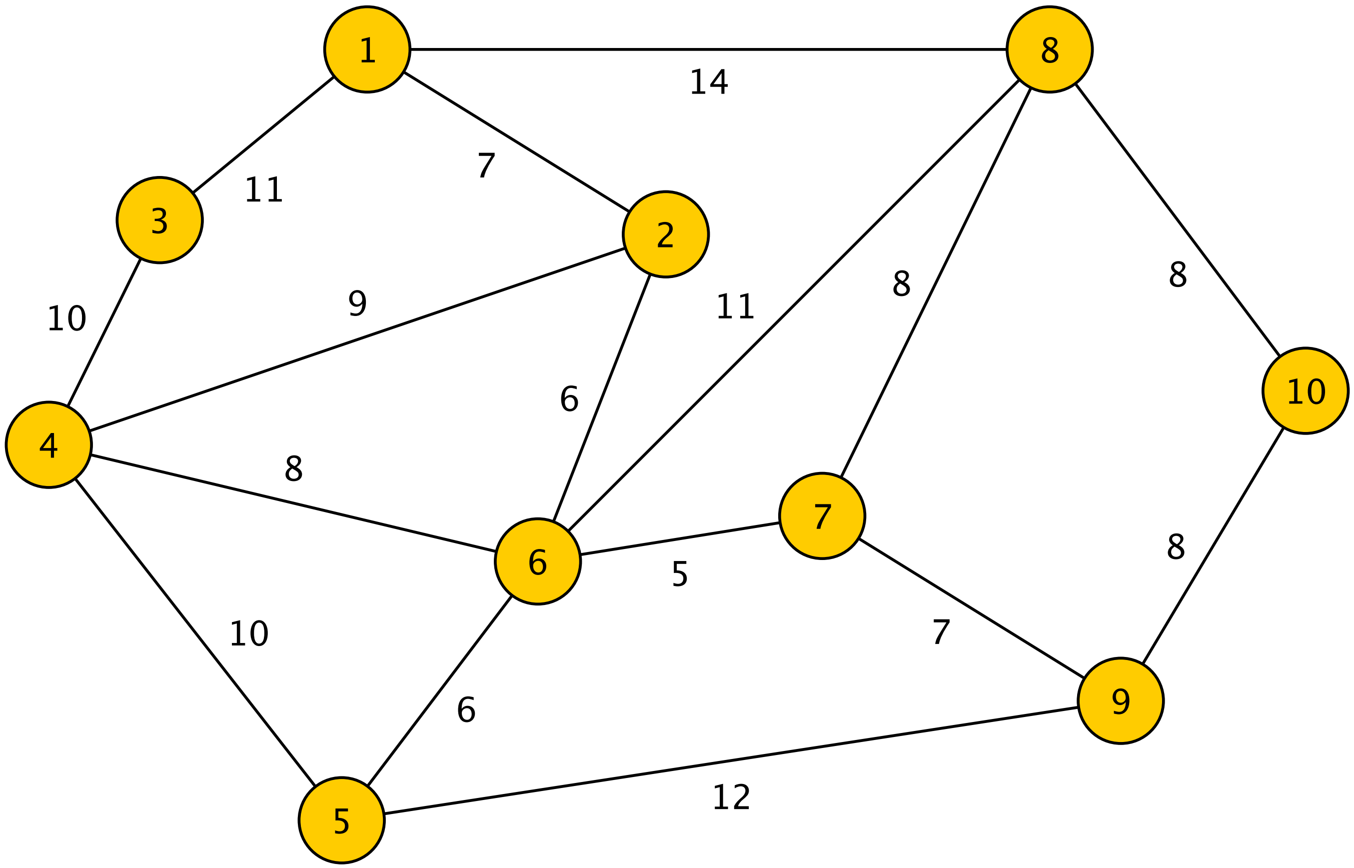
- Assume Map operations are O(1) (for now)
  - |E| = number of edges
  - |V| = number of vertices
- Runtime of add, addEdge, getEdge, removeEdge, remove?
- Space usage?
- Conclusions
  - Matrix is better for dense graphs
  - List is better for sparse graphs
  - For graphs "in the middle" there is no clear winner

Efficiency : Assuming Fast Map

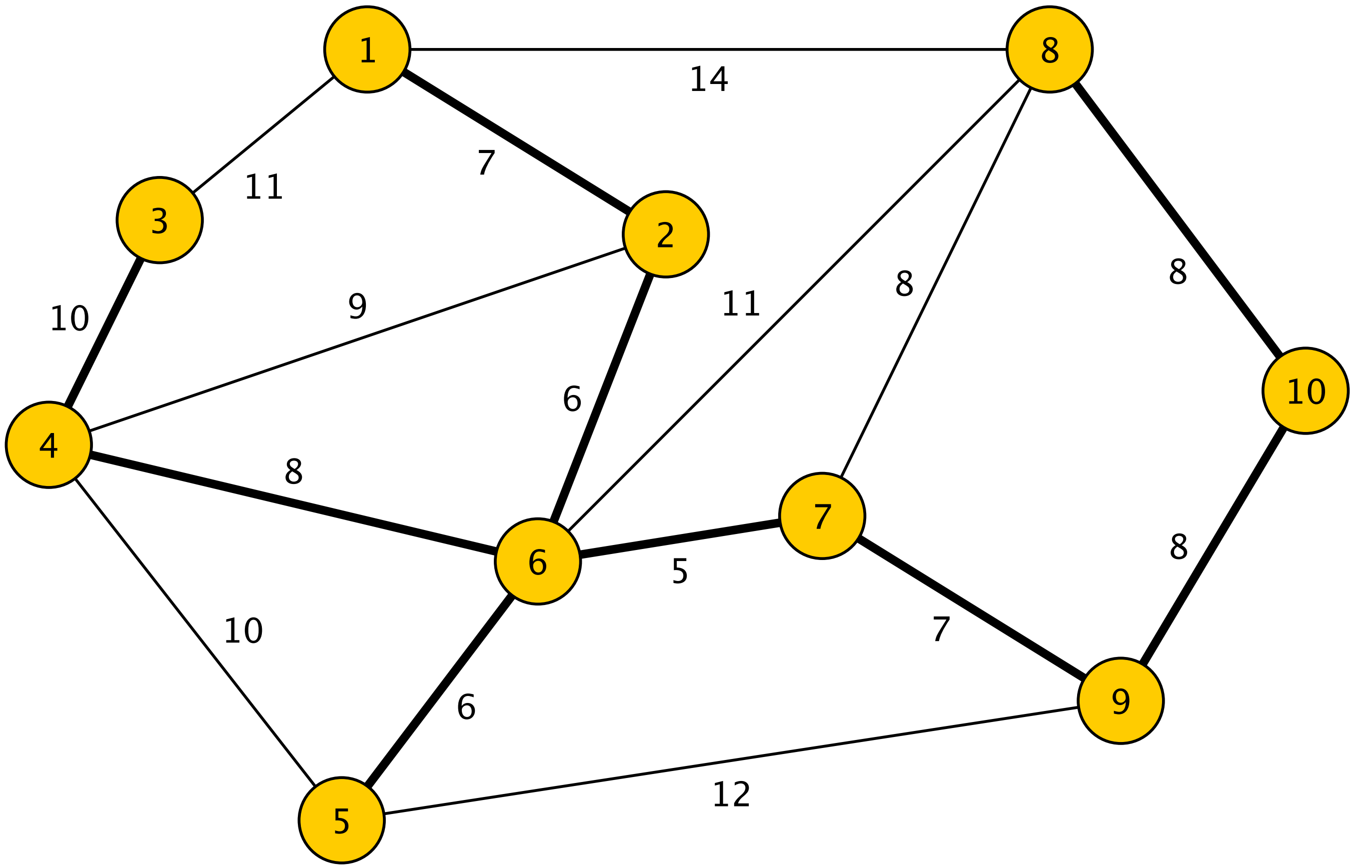
	Matrix	GraphList
add	O(1)	O(1)
addEdge	O(1)	O(1)
getEdge	O(1)	O(|V|)
removeEdge	O(1)	O(|V|)
remove	O(|V|)	O(|V|+|E|)
space	O(|V|²)	O(|V|+|E|)

Applications

Minimum-Cost Spanning Trees



Minimum-Cost Spanning Trees



Basic Graph Properties

- A *subgraph* of a graph G=(V, E) is a graph G'=(V',E') where
  - $V' \subseteq V$
  - $E' \subseteq E$, and
  - If $e \in E'$ where $e = \{u,v\}$, then $u, v \in V'$
- Special Subgraphs
  - If E' contains *every* edge of E having both ends in V', then G' is called the subgraph of G *induced by* V'
  - If V' = V, then G' is called a *spanning subgraph of G*

Basic Graph Properties

- Recall: An undirected graph G=(V,E) is *connected* if for every pair u,v in V, there is a path from u to v (and so from v to u)
- The maximal sized connected subgraphs of G are called its *connected components*
  - Note: They are induced subgraphs of G
- An undirected graph without cycles is a *forest*
- A connected forest is called a *tree.*
  - Not to be confused with the data structure!

Facts About Graphs

Thm: If $G=(V,E)$ is a forest with $|E| > 0$, then G has at least one vertex v of degree 1 (a *leaf*)

- Let's prove this: Consider a longest simple path in G…

Thm: If $G=(V,E)$ is a tree then $|E| = |V| - 1$.

- Hint: Induction on v: delete a leaf

Thm: Every connected graph $G=(V,E)$ contains a spanning subgraph $G'=(V,E')$ that is a tree

- That is, a *spanning tree*

Proof idea:

- If G is not a tree, then it contains a cycle C
- Removing an edge from C leaves G connected (why)
- Repeat until no more cycles remain

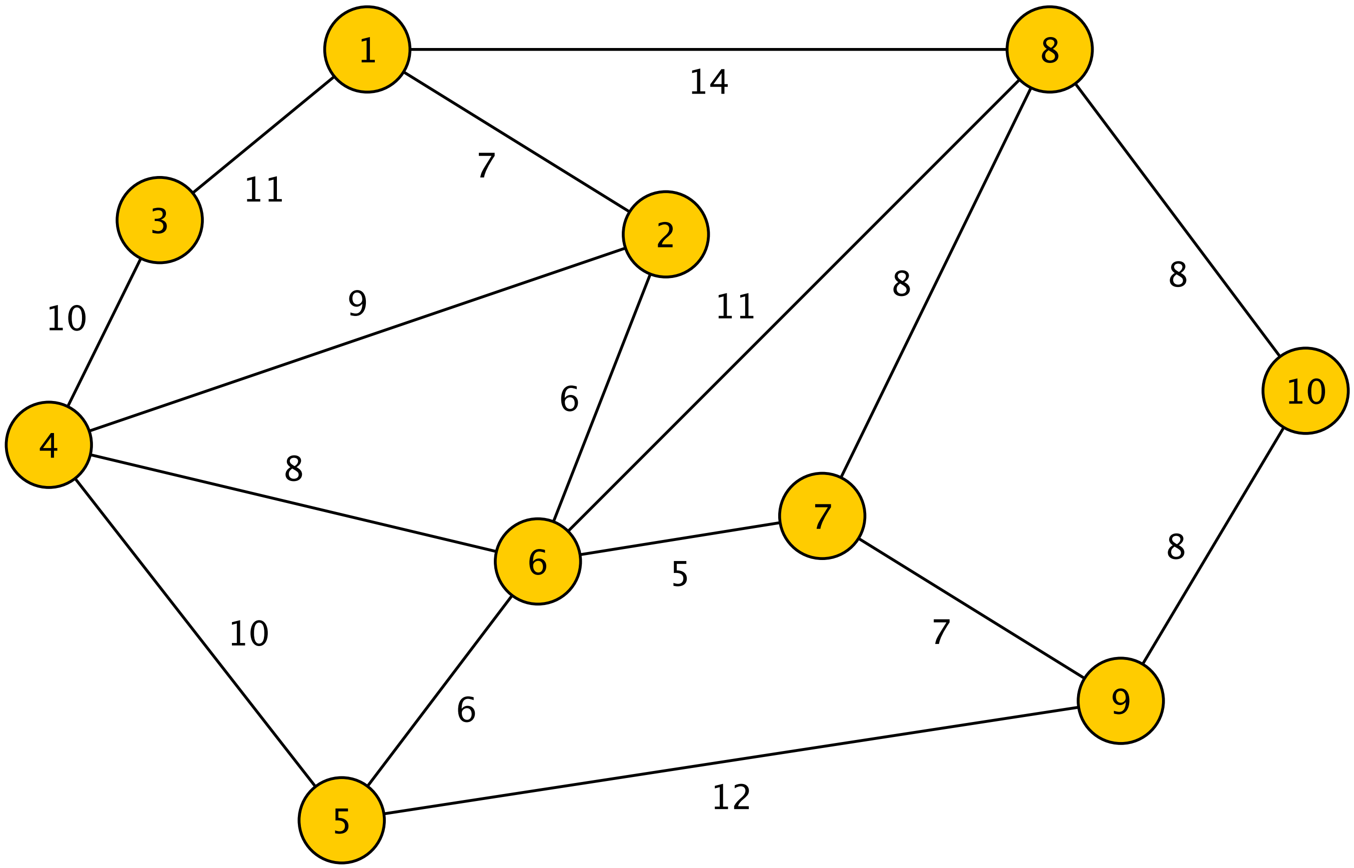
Edge-Weighted Graphs

- An *edge-weighting* of a graph G=(V,E) is an assignment of a number (weight) to each edge of G
  - We write the weight of e as w(e) or w_e
- The weight w(G') of any subgraph G' of G is the sum of the weights of the edges in G'
- We will focus on edge-weights that are non-negative, so if G' is a subgraph of G, then w(G') $\leq$ w(G)

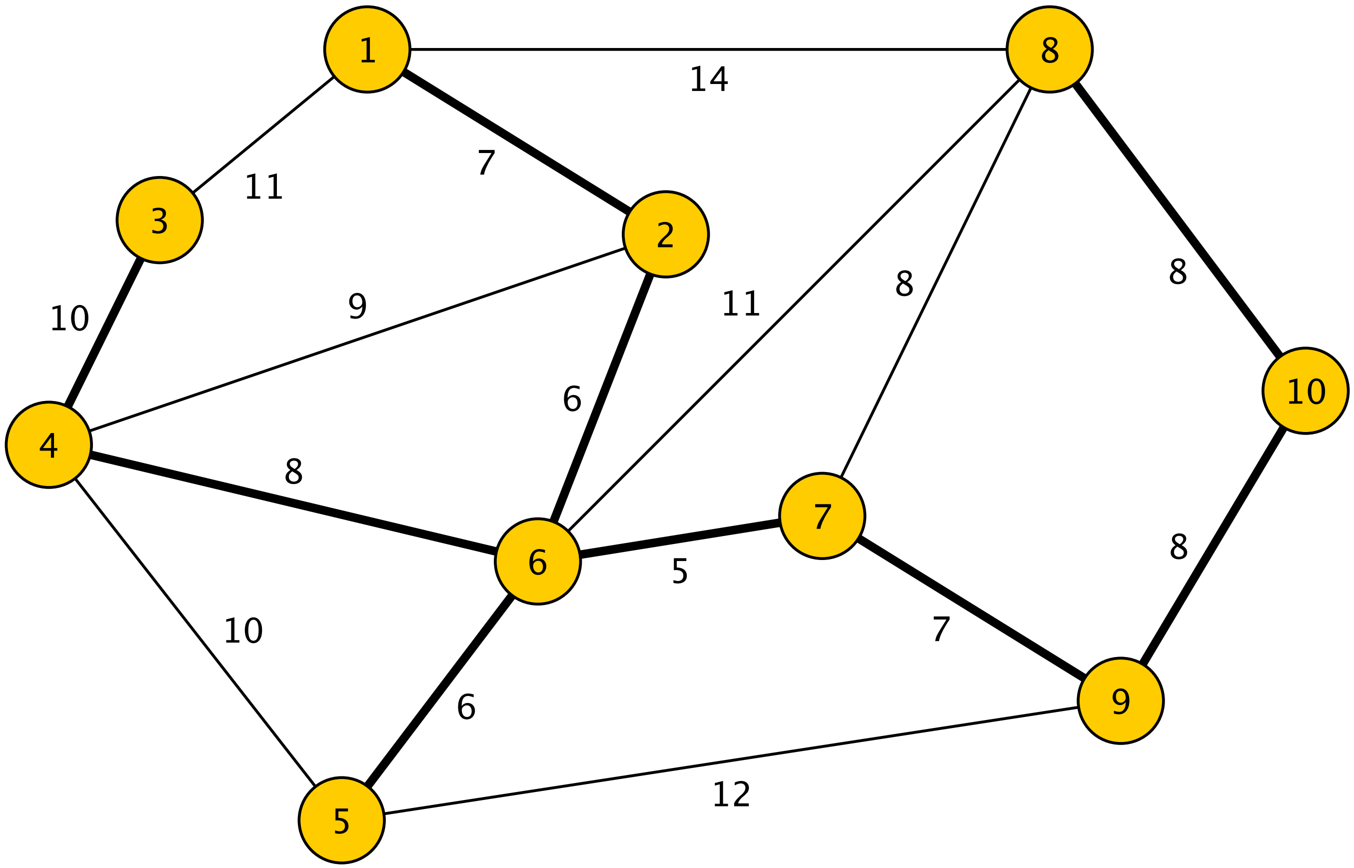
A Famous Problem

- Given a connected, undirected graph $G=(V,E)$ with non-negative edge weights, find a minimum-weight, connected, spanning subgraph of G.
- Note: Such a subgraph must be a spanning tree!
- Frequently, we refer to the edge weights as *costs* and so this problem becomes:
- Given an undirected graph G with edge costs, compute a minimum-cost spanning tree of G.

Minimum-Cost Spanning Trees



Minimum-Cost Spanning Trees



Finding a MCST

Suppose we just wanted to find a PCST (pretty cheap spanning tree), here's one idea:

Grow It Greedily!

- Pick a vertex and find its cheapest incident edge. Now we have a (small) tree
- Repeatedly add the cheapest edge to the tree that keeps it a tree (connected, no cycles)
- This method is called *Prim's Algorithm*
- How close might this get us to the MCST?

An Amazing Fact

Thm: (Prim 1957) The greedy tree-growing algorithm always finds a minimum-cost spanning tree for any connected graph.

Contrast this with the greedy exam scheduling algorithm, which does *not* always find a minimum schedule (coloring)

Why does this work?

The Key

Def: Sets V_1 and V_2 form a *partition* of a set V if

$$V_1 \cup V_2 = V \text{ and } V_1 \cap V_2 = \emptyset$$

Lemma: Let G=(V,E) be a connected graph and let V_1 and V_2 be a partition of V. *Every* MCST of G contains a cheapest edge between V_1 and V_2

- Let e be a cheapest edge between V_1 and V_2
- Let T be a MCST of G. If $e \notin T$, then $T \cup \{e\}$ contains a cycle C and e is an edge of C
- Some other edge e' of C must also be between V_1 and V_2; e is a cheapest edge, so w(e') = w(e) [Why?]