Problem Set 2

Instructions: I encourage you to do all of these problems, but please hand in only the ones labeled "Hand In"!

You might find the *Second Principle of Mathematical Induction* handout useful to read before attempting these problems!

Second Principle of Induction

(*3 points*) . . . . Hand In

Prove that every integer $n \geq 8$ can be written as a sum of 3s and 5s.

(*4 points*) . . . . Hand In

Define a sequence of numbers $t_1, \ldots, t_n, \ldots$ as follows:

1. $t_1 = 1$
2. For all $n \geq 2$, if n is even, then $t_n = 2t_{n/2}$; if n is odd, $t_n = 2t_{(n-1)/2}$.

So $t_2 = 2t_1 = 2, t_3 = 2t_1 = 2, t_4 = 2t_2 = 4, t_5 = 2t_2 = 4, t_6 = 2t_3 = 4, \ldots$.

Prove that for all $n \geq 1$, $t_n \leq n$. Hint: In the induction step, consider 2 cases: $n + 1$ is even and $n + 1$ is odd.

(*4 points*) . . . . Hand In

A string s of left and right "square" and/or "round" parentheses is *balanced* if,

1. `s` is the empty string (`s = ""`),
2. `s = "(" + t + ")"`, where `t` is a balanced string of parentheses,
3. `s = "[" + t + "]"`, where `t` is a balanced string of parentheses, or
4. `s = t + u`, where `t` and `u` are balanced strings of parentheses each shorter than `s`.

Note: In the definition above, + is the string concatenation operator. So, for example, "()", "[]", "[()[]]", "([])[[]()]", are all balanced strings, but "[()[(])]" is not.

Prove by induction if a non-empty string is balanced it must contain as a substring either "()" or "[]".* You can assume that all strings are even length. Hint: Here your induction step must consider 3 cases!

(*0 points*) . . . . Practice

Prove that every integer $n > 1$ can be written as a product of prime numbers. Hint: Use the fact proven in the handout that every $n > 1$ has a prime factor as the basis for an induction proof. Could you have solved this problem as easily using the First Principle of Induction?

(*0 points*) . . . . Practice—Challenge!

A *triomino* (pronounced like "try" followed by "dominoes" without the "d") is a shape made of three unit squares in an "L"-shape. Think of it as the result of removing a single square from a 2×2-checkerboard. If I remove *any* single square from a 4×4 checkerboard, I can cover the remaining 15 squares with 5 non-overlapping tri-ominoes. Try it and see!

Prove by induction that if *any* square is removed from a $2^n \times 2^n$-checkerboard, then the remaining squares can be covered by non-overlapping tri-ominoes. I love this problem: The proof is very short but requires just the right induction step!

*Note: We are finally verifying that fact that you used in your recursion lab last week!.