The Second Principle of Mathematical Induction

CSCI 136: Fall 2018
Handout 4
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More on Induction

In a prior handout, *Induction Essentials*, several versions of the First Principle of Mathematical Induction (sometimes called *weak induction*) were introduced. In the weak form, however, often this principle cannot be directly applied. To see this, consider the following problem.

Example Prove that every integer $n > 1$ has a *prime factor* p; that is, p is a prime* number and $n = p * q$ for some integer q.

Note: If $n = p * q$ for integers p and q, we say that p *divides* n. [Clearly q then also divides n.]

Let's check some cases: 2 and 3 are themselves prime; $4 = 2 \cdot 2$, 5 is prime, $6 = 2 \cdot 3$. So far, so good. Let's try a proof by induction.

Base Cases: We've done several above.
Induction Hypothesis: For some $n \geq 2$, n has a prime factor.
Induction Step: Show, using the Induction Hypothesis, that $n + 1$ has a prime factor.
What do we do now? There's no helpful relationship between numbers that divide n and those that divide $n + 1$. The only thing we can say is that either $n+1$ itself is prime (Yay! We'd be done!) or that $n+1$ has a factor k, $1 < k < n+1$; that is, k divides n. However, we don't know that k is prime. It would be great if we could say that "by induction" k has a prime factor p, since then p would also be a factor of n. But we can't use our First Principle of Induction on p, since p may be *much less* than n.
The Second Principle of Mathematical Induction, described in class, comes to our rescue.

Theorem 1 (The Second Principle of Mathematical Induction (Strong Induction))**.** *Let $P_0, P_1, \ldots, P_n, \ldots$ be a sequence of propositions, one for each integer $n \geq 0$. Suppose that, for some $b \geq 0$*

- *$P_0, P_1, \ldots, P_b$ are true, and that*
- *For every $n \geq b$, if $P_0, P_1, \ldots, P_n$ are all true, then P_{n+1} is true.*
  Note: This condition is often written as: For every $n \geq b$, if P_k is true for all $k : 0 \leq k \leq n$, then P_{n+1} is true.

Then all of the propositions $P_0, P_1, \ldots, P_n, \ldots$ are true.

Note: Starting with P_0 is a convention, but it may be that the first proposition is P_1, P_{17}, or some other value. The theorem is still valid in these situations.

Let's apply this to our problem. Since $1 < k < n + 1$, then k has a prime factor p. But if p divides k, and k divides n, then p divides n † Done!

Let's try another example.

Example

You might remember from high school that the sum of the interior angles of a triangle equals $180°$. Let's prove that the sum of the interior angles of any n-sided polygon is $(n - 2)180°$.

Base Case: $n = 3$. Proved by your high school geometry teacher.

* A positive integer $p > 1$ is prime if the only positive integers that evenly divide it are itself and 1.
† Hopefully this is clear. But just in case: if p divides k than $k = pa$ for some integer a, and if k divides n then $n = kq$ for some integer q. But then $n = (pa)q = p(aq)$ and aq is an integer, so p divides n.

Induction Hypothesis: For some $n \geq 3$, for all $k : 3 \leq k \leq n$, any k-sided polygon has the sum of its interior angles equal to $(k-2)180°$.

Induction Step: Show now that any $n+1$-sided polygon hasthe sum of its interior angles equal to $(n+1-2)180° = (n-1)180°$.
This is actually pretty easy. Every polygon has at least one *internal diagonal*, that is, a line segment connecting two of the vertices of the polygon that lies completely inside the polygon. Cutting the polygon in two along this segment gives two new polygons, each of which have fewer than $n+1$ sides. In fact, one of them will have k sides and one will have m sides, where $m+k = n+3$ (the diagonal becomes a side of each new polygon, adding two new sides).

But since k and m are each at least 3 and $m+k = n+3$, it must be that each of k and m is less than $n+1$, so, by (strong) induction, the k-sided polygon has its interior angles add to $(k-2)180°$ and the m-sided polygon has its interior angles add to $(m-2)180°$. Gluing them back together shows that the original $n+1$-sided polygon has its interior angles add to $(k-2)180° + (m-2)180° = (k+m-4)180° = (n+1-2)180°$.

The second principle of induction is so useful that often folks use it even when the first principle suffices and that's fine. Although it might not be obvious, the two are, in fact, equivalent. Let's see an example of this.

Example

Let's show that every integer $n \geq 12$ can be written as the sum of 4s and 5s.

Base Case(s): $12 = 4+4+4, 13 = 4+4+5, 14 = 4+5+5, 15 = 5+5+5$.
Induction Hypothesis: For some $n \geq 12$, k can be written as the sum of 4s and 5s, for all $12 \leq k \leq n$.

Induction Step: show that $n+1$ can be written as the sum of 4s and 5s.

Because we have verified the property for $12, 13, 14, 15$, we can assume that $n+1 \geq 16$. Now consider the number $n+1-4 = n-3$. Since $n+1 \geq 16$, we have $n-3 \geq 12$. Thus $n-3$ can be written as the sum of 4s and 5s. But then clearly so can $(n-3)+4$—which is just $n+1$.