# Lecture 20

## Trees II

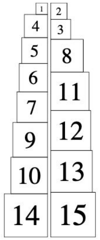
- Lab 6 – Preview
  - Bitwise Operations
  - Gray Code
- Exploring Binary Trees
  - Challenge 1

# Lab 6 – Preview

## 8.7 Laboratory: The Two-Towers Problem

**Objective.** To investigate a difficult problem using `Iterators`.

**Discussion.** Suppose that we are given $n$ uniquely sized cubic blocks and that each block has a face area between 1 and $n$. Build two towers by stacking these blocks. How close can we get the heights of the two towers? The following two towers built by stacking 15 blocks, for example, differ in height by only 129 millions of an inch (each unit is one-tenth of an inch):




Still, this stacking is only the *second-best* solution! To find the best stacking, we could consider all the possible configurations.

We *do* know one thing: the total height of the two towers is computed by summing the heights of all the blocks:

$$h = \sum_{i=1}^{n} \sqrt{i}$$

If we consider all the *subsets* of the $n$ blocks, we can think of the subset as the set of blocks that make up, say, the left tower. We need only keep track of that subset that comes closest to $h/2$ without exceeding it.

In this lab, we will represent a set of $n$ distinct objects by a `Vector`, and we will construct an `Iterator` that returns each of the $2^n$ subsets.

**Computer Science CS136 (Fall 2021)**

*Duane Bailey & Aaron Williams*

Laboratory 6

*The Two Towers Problem*

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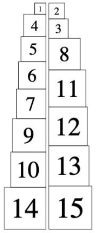


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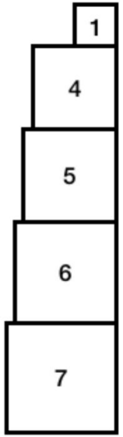
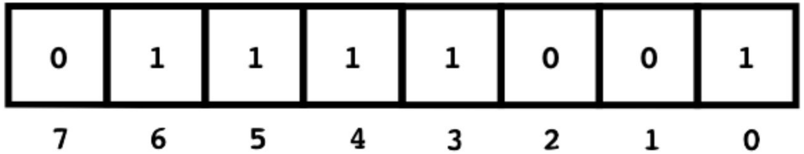
In this lab, we will represent a set of $n$ distinct objects by a `Vector`, and we will construct an `Iterator` that returns each of the $2^n$ subsets.

This lab (right) is a version of the classic Two Towers Problem from the textbook (left). But the extension is new this semester.

Also suppose that we have eight blocks, labeled `1,2,...,8`, and we arbitrarily choose blocks `1, 4, 5, 6,` and `7` as our subset:




If we filled in the corresponding boxes in for each chosen block (i.e., if the block `i` is chosen, we put a `1` in spot `i-1` since our blocks are 1-indexed), our paper strip looks like this:




Conveniently, we can think of the combination of 0s and 1s on strip of paper as the digits of an `n`-bit binary number. For example, the binary number `01111001` is the decimal number `121`, since `01111001` is $2^0 + 2^3 + 2^4 + 2^5 + 2^6$.

https://williams-cs.github.io/cs136-s21-www/labs/two-towers.html

(This link provides some helpful illustrations, but is <u>not</u> the lab handout for this semester.

Each possible solution can be represented as a subset or a binary string or as an integer!

- Integer 121 ⇒ binary string 01111001 ⇒ subset {7, 6, 5, 4, 1}.

# Bitwise Operations

# Bitwise Operations: Isolating a Bit

How can you test if a particular bit is set to `0` or `1` within the binary representation of an integer?

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 |

For example, in the integer `121` is the 5th bit `1` or `0`?

- An `int` in Java has 32 bits (4 bytes) and we index them as $b_{31}\ldots b_1 b_0$ or just $b_7\ldots b_1 b_0$ in a byte.

This is also known as *isolating* the bit.

The 5th bit in `121` is `1`.
But how can we determine this programmatically?

Each bit is worth a power of `2`. More specifically, the $i^{th}$ bit has value $2^i$. For example,

$$
\begin{aligned}
121 &= 0\cdot2^7+1\cdot2^6+1\cdot2^5+1\cdot2^4+1\cdot2^3+0\cdot2^2+0\cdot2^1+1\cdot2^0\\
&= 2^6 + 2^5 + 2^4 + 2^3 + 2^0\\
&= 64 + 32 + 16 + 8 + 1
\end{aligned}
$$

In particular, the 5th bit is worth $2^5 = 32$. In other words, $32_{10} = 00100000_2$. So we isolate it by AND (&) with `32`.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 |

AND (&)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |

=

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |

To obtain `32` we could use repeated multiplication by 2.
Or we can use *bit-shifting* which move bits to the left or right.
For example, `1 << 5` gives `32` in Java, where `1` is the value that is shifted and `5` is the number of positions to shift its bits.

$121 \,\&\, (1 << 5) = 121 \,\&\, 32 = 32 \neq 0$
so the 5th bit is 1.

# Extension

URGENT

Dear Computer Science Majors,

My musical *In Transit* is opening on Broadway **tonight**!

Just this morning I learned that my 90 minute show needs to be divided exactly into two 45 minute halves. (Couldn't they just *reschedule* that fire alarm test?!)

I have contacted everyone that I know and they have not been able to divide the shows by hand. Someone suggested that this could be a Computer Science problem — something about algorithms and complexity? — so I am turning to you.

As a Williams College alum I know that your education has been outstanding, so I know that you can do this.

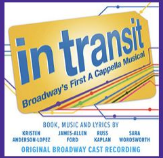
The list of performances and their times is attached. The performances can be reordered however you like, but they must divide into two 45 minute halves.

Thank you so much! Go Ephs!

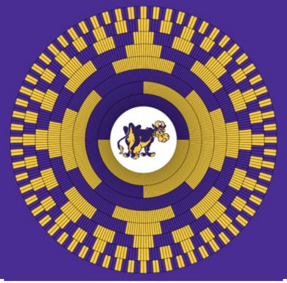
Sincerely,

Kristen Anderson-Lopez

Class of 94!



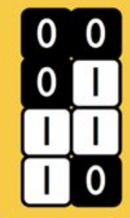

| | |
|---|---|
| "Deep Beneath the City" | 193 |
| "Not There Yet" | 131 |
| "Do What I Do" | 180 |
| "Four Days Home" | 277 |
| "Broke" | 345 |
| "Saturday Night Obsession" | 234 |
| "Wingman" | 454 |
| "But, Ya Know" | 264 |
| "Not There Yet (Reprise)" | 377 |
| "Keep It Goin'" | 178 |
| "A Little Friendly Advice" | 372 |
| "Choosing Not to Know" | 330 |
| "The Moving Song" | 299 |
| "We Are Home" | 434 |
| "Getting There" | 520 |
| "Finale" | 812 |
| | 5400 sec (90 min) |



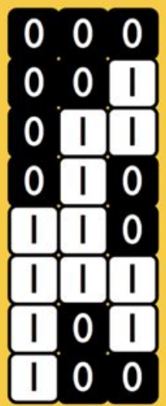
# Gray Code

A Gray Code has the following properties:

- Every n-bit binary string is listed exactly once.
- Consecutive strings differ in exactly one bit.




A Gray code for the 2-bit binary strings.




A Gray code for the 3-bit binary strings.

**Activity**

Arrange the 4-bit binary strings as a Gray code.

**Questions**

- Is your Gray code cyclic?
- How could you build a Gray code for any n?

The extension involves using a Gray code to speed up the exhaustive computation.

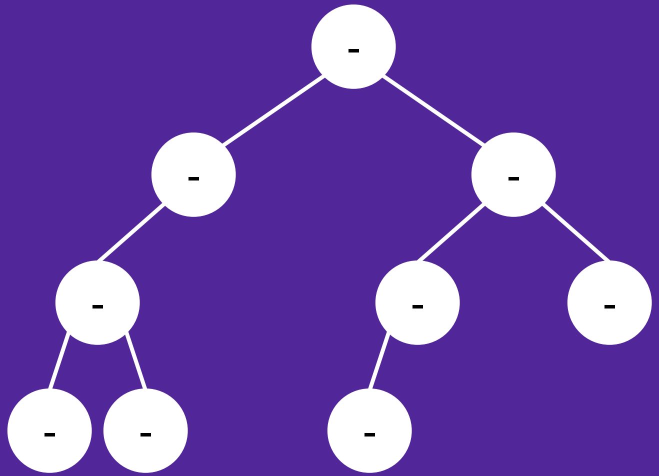
We'll take a 15 minute diversion into Gray codes. It is posted as `20-gray.pdf` and is <u>not</u> testable material.

# Exploring Binary Trees

# Challenge: Exploring Binary Trees (Part 1)

How can we count the number of leaves in a binary tree?

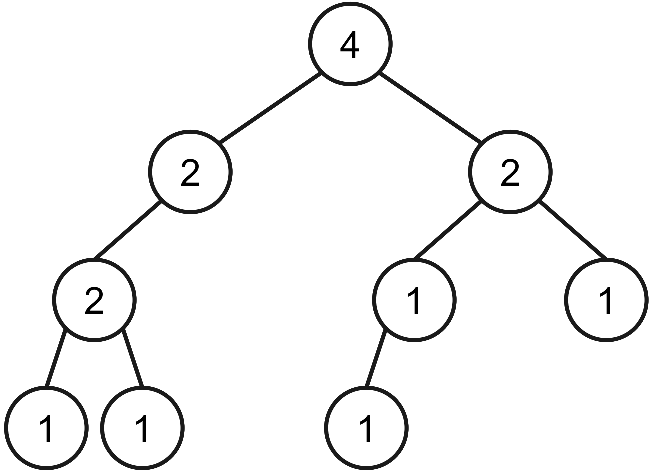
Think about this for 1 minute.
Then discuss it with your neighbor for 2 minutes.




This binary tree has 4 leaves.

Hints:

- Consider a recursive algorithm.
- What are the base cases?




```
// Count the number of leaves.
numLeaves(node)
  // Base case: the root is null
  if node is null then
     return 0

  // Base case: the root is a leaf
  if node.left is null and node.right is null then
    return 1

  // Count leaves in the left and right subtrees.
  numLeft  = numLeaves(node.left)
  numRight = numLeaves(node.right)

  // Return the total.
  return numLeft + numRight

// Main method: Run the algorithm from the tree's root.
answer = numLeaves(root)
```

Counting the number of leaves.

Note: We can visualize the return values in the nodes.

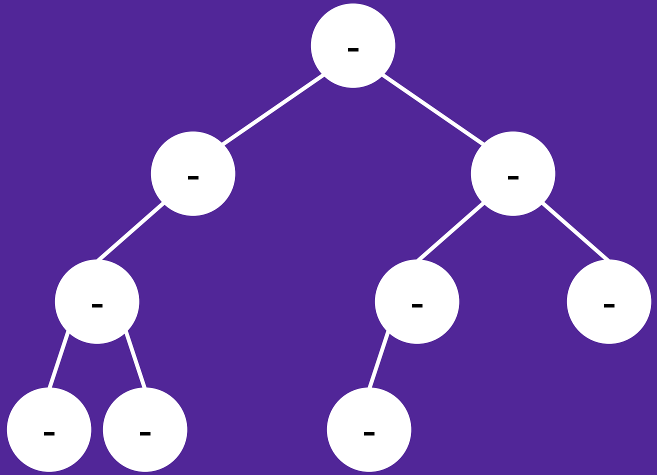
- What order would these values be filled in during the algorithm?

# Challenge: Exploring Binary Trees (Part 2)

How can we determine the following values in a binary tree?

- The total number of nodes.
- The height of the tree.
- The smallest level that has a leaf.
- The number of left links that are used.

Think about this for 1 minute.
Then discuss it with your neighbor for 4 minutes.




This binary tree has 8 total nodes.
It has height 3 (counting from 0).
The smallest level of a leaf is 2.
It has 5 left links in total.

What other quantities could we try to count?

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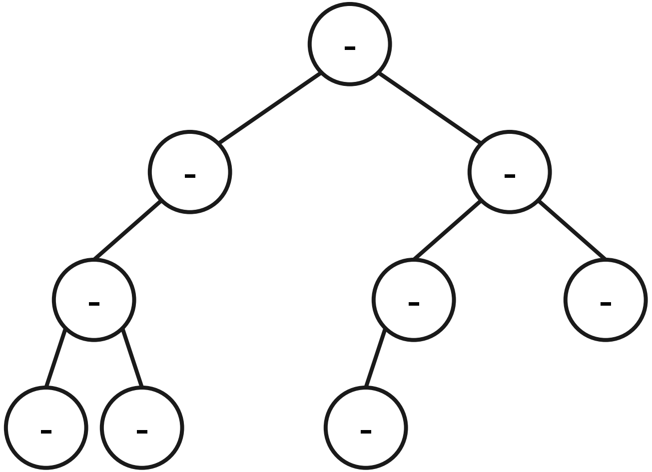
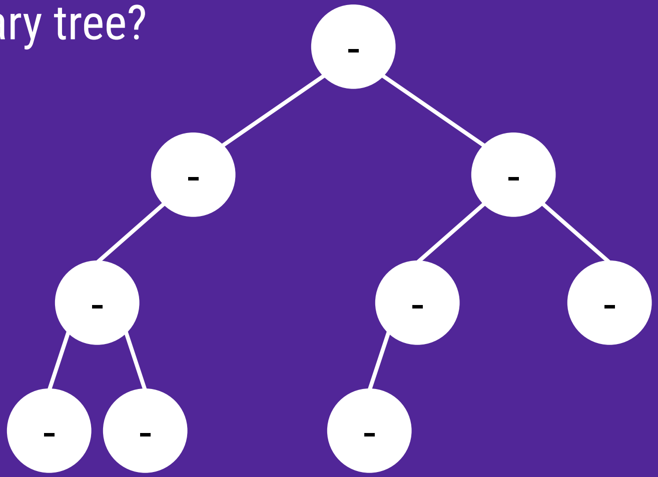



```
// TODO
height(root)
  // Base case:
  if then
     return
```

# Challenge: Exploring Binary Trees (Part 3)

How could we print out a nice text representation of a binary tree?

Think about this for 1 minute.




This binary tree could be printed out as

```
   o
  / \
  o  o
 /  /\
 o  o o
/\ /
o oo
```

Questions:

- What do you interpret *nice* to mean?
- What values would you want to compute?