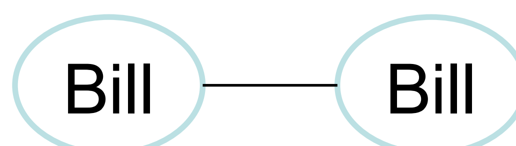
CSCI 136
Data Structures & Advanced Programming

Lecture 27

Fall 2018

Instructors: 

Last Time

- Introduction To Graphs
  - Definitions and Properties: Undirected Graphs

Today's Outline

- More on Graphs
  - Applications and Problems
    - Testing connectedness
    - Counting connected components
      - Breadth-first and Depth-first search
  - Directed Graphs
    - Definition and Properties
  - Reachability and (Strong) Connectedness
- Graph Data Structures: Preliminaries
  - Graph Interface

Reachability and Connectedness

- Def'n: A vertex v in G is *reachable* from a vertex u in G if there is a path from u to v
- v is reachable from u *iff* u is reachable from v
- Def'n: An undirected graph G is *connected* if for every pair of vertices u, v in G, v is reachable from u (and vice versa)
- The set of all vertices reachable from v, along with all edges of G connecting any two of them, is called the *connected component of v*

Basic Graph Algorithms

- We'll look at a number of graph algorithms
  - Connectedness: Is G connected?
    - If not, how many *connected components* does G have?
  - Cycle testing: Does G contain a cycle?
    - Does G contain a cycle through a given vertex?
  - If the edges of G have costs:
    - What is the cheapest subgraph connecting all vertices
      - Called a *connected, spanning subgraph*
    - What is a cheapest path from u to v?
  - And more….

Operations on Graphs

- What are the basic operations we need to describe algorithms on graphs?
  - Given vertices u and v: are they *adjacent*?
  - Given vertex v and edge e, are they *incident*?
  - Given an edge e, get its incident vertices (*ends*)
  - How many vertices are adjacent to v? (*degree* of v)
    - The vertices adjacent to v are called its *neighbors*
  - Get a list of the vertices *adjacent* to v
    - From which we can get the edges *incident* with v

Testing Connectedness

- How can we determine whether G is connected?
  - Pick a vertex v; see if every vertex u is reachable from v
- How could we do this?
  - Visit the neighbors of v, then visit their neighbors, etc. See if we reach all vertices
    - Assume we can mark a vertex as "visited"
- How do we manage all of this visiting?
  - Let's try an example…

Reachability: Breadth-First Search

BFS(G, v)   *// Do a breadth-first search of G starting at v*

// pre: all vertices are marked as unvisited

count ← 0;

Create empty queue Q; enqueue v; mark v as visited; count++

While Q isn't empty

    current ← Q.dequeue();

    for each unvisited neighbor u of current :

        add u to Q; mark u as visited; count++

return count;

Now compare value returned from BFS(G,v) to size of V

BFS Reflections

- The BFS algorithm traced out a tree T_v: the edges connecting a visited vertex to (as yet) unvisited neighbors
- T_v is called a *BFS tree of G with root v* (or *from v*)
- The vertices of T_v are visited in *level-order*
- This reveals a natural measure of distance between vertices: the length of (any) shortest path between the vertices

Distance in Undirected Graphs

Def: The *distance* between two vertices u and v in an undirected graph $G=(V,E)$ is the minimum of the path lengths over all u-v paths.

- It is the depth of u in T_v: a BFS tree from v
- We write it as $d(u,v)$. It satisfies the properties
  - $d(u,u) = 0$, for all $u \in V$
  - $d(u,v) = d(v,u)$, for all $u,v \in V$
  - $d(u,v) \leq d(u,w) + d(w,v)$, for all $u,v,w \in V$
- This last property is call the *triangle inequality*

Reachability: Depth-First Search

DFS(G, v) *// Do a depth-first search of G starting at v*

// pre: all vertices are marked as unvisited

count ← *0;*

Create empty stack S; push v; mark v as visited; count++;

While S isn't empty

    current ← *S.pop();*

    for each unvisited neighbor u of current :

        add u to S; mark u as visited; count++

return count;

Now compare value returned from DFS(G,v) to size of V

DFS Reflections

- The DFS algorithm traced out a tree different from that produced by BFS
  - It still consists of the edges connecting a visited vertex to (as yet) unvisited neighbors
- It is called a DFS *tree of G with root v* (or *from v*)
- Vertices are processed in pre-order w.r.t. the tree
- By manipulating the stack differently, we could produce a post-order version of DFS
- And perhaps write DFS recursively….

Recursive Depth-First Search

```
// Before first call to DFS, set all vertices to unvisited
//Then call DFS(G,v)
DFS(G, v)
        Mark v as visited; count = 1;
        for each unvisited neighbor u of v:
                count += DFS(G,u);
        return count;
```

Is it even clear that this method does what we want?!

Let's prove some facts about it....

Recursive Depth-First Search

Claim: DFS visits all vertices w reachable from v

•Proof: Induction on length d of shortest path from v to w

- Base case: d = 0: Then v = w ✓
- Ind. Hyp.: Assume DFS visits all vertices w of distance at most d from v (for some $d \geq 0$).
- Ind. Step: Suppose now that w is distance d+1 from v. Consider a path of length d+1 from v to w and let u be the next-to-last vertex on the path

Recursive Depth-First Search

Claim: DFS visits all vertices w reachable from v

- Proof: Induction on length d of shortest path from v to w
  - The path is $v = v_0, v_1, v_2, \ldots, v_d = u, v_{d+1} = w$
    - The edges are implied so not explicitly written!
  - By Ind. Hyp., u is visited. At this point, if w has not yet been visited, it will be one of the unvisited vertices on which DFS() is recursively called, so it will then be visited.

Recursive Depth-First Search

Claim: DFS visits *only* vertices reachable from v

- Idea: Prove by induction on number of times DFS is called that DFS is only called on vertices w reachable from v

Claim: DFS counts correctly the number of vertices reachable from v

- Idea: Induction on number of unvisited vertices reachable from v
  - DFS will never be called on same vertex twice

Recursive Depth-First Search

Claim: DFS(G,v) returns the number of unvisited nodes reachable from v

Proof: Uses previous two observations

- DFS visits every node reachable from v
- DFS doesn't visit any node *not* reachable from v