CSCI 136
Data Structures &
Advanced Programming

Lecture 9

Fall 2017

Instructors: Bills

Administrative Details

- Lab 3 Today!
  - You *may* work with a partner
  - Come to lab with a plan!
  - Try to answer questions before lab

Last Time

- Note: Storing null values in Lists
- More on Doubly-Linked List
  - Lab this week: Doubly Linked Lists with dummy nodes
- Abstract Classes and Inheritance
  - Return of the Card Classes!
- The Structure5 Universe to date

Today

- Measuring Growth
  - Big-O
- Introduction to Recursion

Measuring Computational Cost

Consider these two code fragments…

```
for (int i=0; i < arr.length; i++)
   if (arr[i] == x) return "Found it!";
```

…and…

```
for (int i=0; i < arr.length; i++)
   for (int j=0; j < arr.length; j++)
     if( i !=j && arr[i] == arr[j]) return "Match!";
```

How long does it take to execute each block?

Measuring Computational Cost

- How can we measure the amount of work needed by a computation?
  - Absolute clock time
    - Problems?
      - Different machines have different clocks
      - Too much other stuff happening (network, OS, etc)
      - Not consistent. Need lots of tests to predict future behavior

Measuring Computational Cost

- A better way: Counting computations
  - Count *all* computational steps?
  - Count how many "expensive" operations were performed?
  - Count number of times "x" happens?
    - For a specific event or action "x"
    - i.e., How many times a certain variable changes
- Question: How accurate do we need to be?
  - 64 vs 65? 100 vs 105? Does it really matter??

An Example

```
// Pre: array length n > 0
public static int findPosOfMax(int[] arr) {
        int maxPos = 0 // A wild guess
        for(int i = 1; i < arr.length; i++)
                if (arr[maxPos] < arr[i]) maxPos = i;
        return maxPos;
}
```

- Can we count steps exactly?
  - ”if” makes it hard
- Idea: Overcount: assume “if” block always runs
- Overcounting gives *upper bound* on run time
- Can also undercount for lower bound
- Overcount: 4(n-1) + 4; undercount: 3(n-1) + 4

Measuring Computational Cost

- Rather than keeping exact counts, we want to know the *order of magnitude* of occurrences
  - 60 vs 600 vs 6000, *not* 65 vs 68
  - n, *not* 4(n-1) + 4
- We want to make comparisons without looking at details and without running tests
- Avoid using specific numbers or values
- Look for overall trends

Measuring Computational Cost

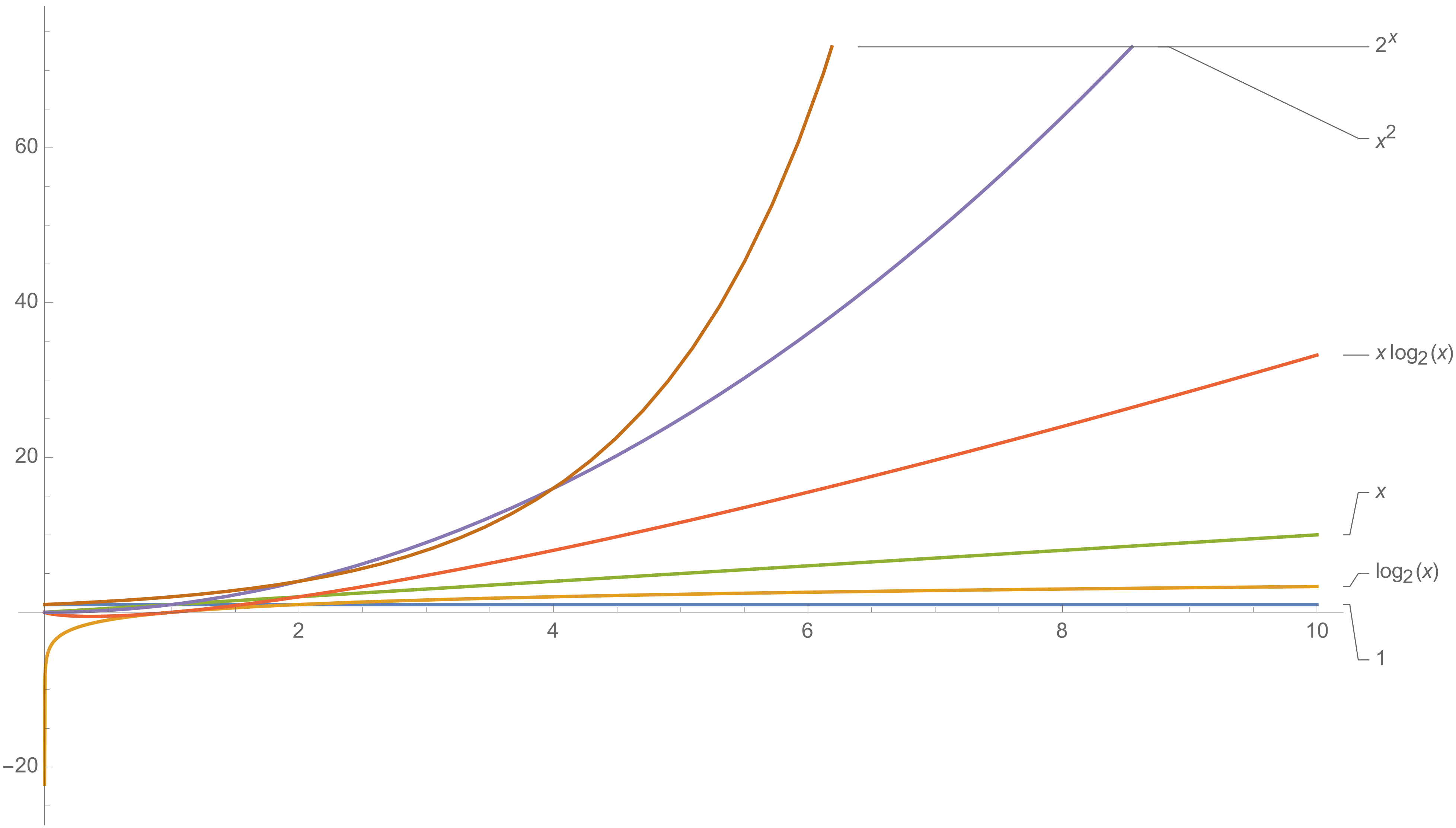
- How does algorithm scale with problem size?
  - E.g.: If I double the size of the problem instance, how much longer will it take to solve:
    - Find maximum: $n - 1 \rightarrow (2n) - 1$ (≈ twice as long)
    - Bubble sort: $n(n-1)/2 \rightarrow 2n(2n - 1)/2$ (≈ 4 times as long)
    - Subset sum: $2^{n-1} \rightarrow 2^{2n-1}$ (2^n times as long!!!)
    - Etc.
- We will also measure amount of space used by an algorithm using the same ideas….

Function Growth

Consider the following functions, for $x \geq 1$

- $f(x) = 1$
- $g(x) = \log_2(x)$ // Reminder: if x=2^n, $\log_2(x) = n$
- $h(x) = x$
- $m(x) = x \log_2(x)$
- $n(x) = x^2$
- $p(x) = x^3$
- $r(x) = 2^x$

Function Growth



Function Growth & Big-O

- Rule of thumb: ignore multiplicative constants
- Examples:
  - Treat n and n/2 as same order of magnitude
  - $n^2/1000$, $2n^2$, and $1000n^2$ are "pretty much" just n^2
  - $a_0n^k + a_1n^{k-1} + a_2n^{k-2} + \cdots a_k$ is roughly n^k
- The key is to find the most *significant* or *dominant* term
- Ex: $\lim_{x\to\infty} (3x^4 - 10x^3 - 1)/x^4 = 3$ (Why?)
  - So $3x^4 - 10x^3 - 1$ grows "like" x^4

Asymptotic Bounds (Big-O Analysis)

- A function f(n) is *O(g(n))* if and only if there exist positive constants c and n_0 such that

$$|f(n)| \leq c \cdot g(n) \text{ for all } n \geq n_0$$

- g is "at least as big as" f **for large n**
  - Up to a multaplicative constant c!
- Example:
  - $f(n) = n^2/2$ is $O(n^2)$
  - $f(n) = 1000n^3$ is $O(n^3)$
  - $f(n) = n/2$ is $O(n)$

Determining "Best" Upper Bounds

- We typically want the *smallest* upper bound when we estimate running time
- Example: Let $f(n) = 3n^2$
  - $f(n)$ is $O(n^2)$
  - $f(n)$ is $O(n^3)$
  - $f(n)$ is $O(2^n)$ (see next slide)
  - $f(n)$ is NOT $O(n)$ (!!)
- "Best" upper bound is $O(n^2)$
- We care about **c** and $\mathbf{n_0}$ in practice, but focus on size of **g** when designing algorithms and data structures

What's n_0? Messy Functions

- Example: Let $f(n) = 3n^2 - 4n + 1$. $f(n)$ is $O(n^2)$
  - Well, $3n^2 - 4n + 1 \le 3n^2 + 1 \le 4n^2$, for $n \ge 1$
  - So, for $c = 4$ and $n_0 = 1$, we satisfy Big-O definition
- Example: Let $f(n) = n^k$, for any fixed $k \ge 1$. $f(n)$ is $O(2^n)$
  - Harder to show: Is $n^k \le c\ 2^n$ for some $c > 0$ and large enough n?
  - It is if and only if $\log_2(n^k) \le \log_2(2^n)$, that is, iff $k \log_2(n) \le n$.
  - That is iff $k \le n/\log_2(n)$. But $n/\log_2(n) \rightarrow \infty$ as $n \rightarrow \infty$
  - This implies that for some n_0 on $n/\log_2(n) \ge k$ if $n \ge n_0$
  - Thus $n \ge k \log_2(n)$ for $n \ge n_0$ and so $2^n \ge n^k$

Input-dependent Running Times

- Algorithms may have different running times for different input values
- Best case (typically not useful)
  - Sort already sorted array in O(n)
  - Find item in first place that we look O(1)
- Worst case (generally useful, sometimes misleading)
  - Don’t find item in list O(n)
  - Reverse order sort $O(n^2)$
- Average case (useful, but often hard to compute)
  - Linear search O(n)
  - QuickSort random array O(n log n) ← We’ll sort soon

Vector Operations : Worst-Case

For n = Vector size (*not* capacity!):

- O(1): size(), capacity(), isEmpty(), get(i), set(i), firstElement(), lastElement()
- O(n): indexOf(), contains(), remove(elt), remove(i)
- What about add methods?
  - If Vector doesn't need to grow
    - add(elt) is O(1) but add(elt, i) is O(n)
  - Otherwise, depends on ensureCapacity() time
    - Time to compute newLength : $O(\log_2(n))$
    - Time to copy array: O(n)
    - $O(\log_2(n)) + O(n)$ is O(n)

Vectors: Add Method Complexity

Suppose we grow the Vector's array by a fixed amount d.

How long does it take to add n items to an empty Vector?

- The array will be copied each time its capacity needs to exceed a multiple of d
  - At sizes 0, d, 2d, … , n/d.
- Copying an array of size kd takes ckd steps for some constant c, giving a total of

$$\sum_{k=1}^{n/d} ckd = cd \sum_{k=1}^{n/d} k = cd \left(\frac{n}{d}\right)\left(\frac{n}{d} + 1\right)/2 = O(n^2)$$

Vectors: Add Method Complexity

Suppose we grow the Vector's array by doubling.

How long does it take to add n items to an empty Vector?

- The array will be copied each time its capacity needs to exceed a power of 2
  - At sizes 0, 1, 2, 4, 8 … $2^{\log_2 n}$
- Copying an array of size 2^k takes c 2^k steps for some constant c, giving a total of

$$\sum_{k=1}^{\log_2 n} c2^k = c \sum_{k=1}^{\log_2 n} 2^k = c\,(2^{\log_2 n+1} - 1) = O(n)$$

- Very cool!