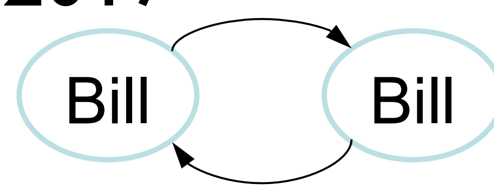
CSCI 136
Data Structures &
Advanced Programming

Lecture 28

Fall 2017

Instructors:



Last Time

- More on Graphs
  - Built up a vocabulary to talk about graphs
  - Proved some things about graphs
  - Introduced:
    - Connectedness
    - Reachability

This Time

- More on Graphs
  - Applications and Problems
    - Testing connectedness
    - Counting connected components
    - Breadth-first
    - Depth-first search
      - And recursive depth-first search
  - Directed Graphs : Introduction

Next Time?

- More Directed Graphs
  - Reachability and (Strong) Connectedness
- Graph Data Structures: Implementation
  - Graph Interface
  - Adjacency Array Implementation Basic Concept
  - Adjacency List Implementation Basic Concept
  - Adjacency Array Implementation Details

Basic Graph Algorithms

- We'll look at a number of graph algorithms
  - Connectedness: Is G connected?
    - If not, how many *connected components* does G have?
  - Cycle testing: Does G contain a cycle?
    - Does G contain a cycle through a given vertex?
  - If the edges of G have costs:
    - What is the cheapest subgraph connecting all vertices
      - Called a *connected, spanning subgraph*
    - What is a cheapest path from u to v?
  - And more....

Operations on Graphs

- What are the basic operations we need to describe algorithms on graphs?
  - Given vertices u and v: are they *adjacent*?
  - Given vertex v and edge e, are they *incident*?
  - Given an edge e, get its incident vertices (*ends*)
  - How many vertices are adjacent to v? (*degree* of v)
    - The vertices adjacent to v are called its *neighbors*
  - Get a list of the vertices *adjacent* to v
    - From which we can get the edges *incident* with v

Testing Connectedness

- How can we determine whether G is connected?
  - Pick a vertex v; see if every vertex u is reachable from v
- How could we do this?
  - Visit the neighbors of v, then visit their neighbors, etc. See if you reach all vertices
    - Assume we can mark a vertex as “visited”
- How do we manage all of this visiting?
  - Let’s try an example…

Reachability: Breadth-First Search

```
BFS(G, v)        // Do a breadth-first search of G starting at v
// pre: all vertices are marked as unvisited
count ←0;
Create empty queue Q; enqueue v; mark v as visited; count++
While Q isn't empty
        current ←Q.dequeue();
        for each unvisited neighbor u  of current :
                add u to Q; mark u as visited; count++
return count;
```

Now compare value returned from `BFS(G,v)` to |V|

BFS Reflections

- The BFS algorithm traced out a tree T_v: the edges connecting a visited vertex to (as yet) unvisited neighbors
- T_v is called a *BFS tree* of G with root v
- The vertices of T_v are visited in *level-order*
- This reveals a natural measure of distance between vertices: the length of (any) shortest path between the vertices

Distance in Undirected Graphs

Definition: The *distance* between two vertices `u` and `v` in an undirected graph `G=(V,E)` is the minimum of the path lengths over all `u–v` paths.

- Distance is the depth of `u` in T_v (a BFS tree from `v`)
  - We write distance as `d(u,v)`

Distance in Undirected Graphs

- Distance satisfies the following properties:
  - $d(u,u) = 0$, for all $u \in V$
  - $d(u,v) = d(v,u)$, for all $u,v \in V$
  - $d(u,v) \leq d(u,w)+d(w,v)$, for all $u,v,w \in V$

- The last property is call the *triangle inequality*

Reachability: Depth-First Search

```
DFS(G, v)          // Do a depth-first search of G starting at v
// pre: all vertices are marked as unvisited
count ← 0;
Create empty stack S; push v; mark v as visited; count++;
While S isn't empty
        current ← S.pop();
        for each unvisited neighbor u of current :
                add u to S; mark u as visited; count++
return count;
```

Now compare value returned from `DFS(G,v)` to $|V|$

DFS Reflections

- The DFS algorithm traced out a tree different from that produced by BFS
  - It still consists of the edges connecting a visited vertex to (as yet) unvisited neighbors
- It is called a *DFS tree* of G with root v
- Vertices are visited in pre-order w.r.t. the tree
- By manipulating the stack differently, we could produce a post-order version of DFS
- And perhaps write DFS recursively….

Recursive Depth-First Search

```
// Before first call to DFS, set all vertices to unvisited
//Then call DFS(G,v)
DFS(G, v)
        Mark v as visited; count = 1;
        for each unvisited neighbor u of v:
                count += DFS(G,u);
        return count;
```

Is it even clear that this method does what we want?!

Let's prove some facts about it....

Recursive Depth-First Search

Claim: DFS visits all vertices `w` reachable from `v`

- Proof: Induction on length `d` of shortest path from `v` to `w`
  - Base case: `d = 0`: Then `v = w` ✓
  - Ind. Hyp.: Assume DFS visits all vertices `w` of distance at most `d` from `v` (for some `d ≥ 0`).
  - Ind. Step: Suppose now that `w` is distance `d+1` from `v`. Consider a path of length `d+1` from `v` to `w` and let `u` be the next-to-last vertex on the path.

Recursive Depth-First Search

Claim: DFS visits all vertices w reachable from v

- Proof: Induction on length `d` of shortest path from `v` to `w`
  - The path is $v = v_0, v_1, v_2, \ldots, v_d = u, v_{d+1} = w$
    - (The edges are implied so not explicitly written!)
  - By Ind. Hyp., `u` is visited. At this point, if `w` has not yet been visited, it will be one of the unvisited vertices on which DFS() is recursively called, so it will then be visited.

Recursive Depth-First Search

Claim: DFS visits *only* vertices reachable from v

- Idea: Prove by induction on number of times DFS is called that DFS is only called on vertices w reachable from v

Claim: DFS counts correctly the number of vertices reachable from v

- Idea: Induction on number of unvisited vertices reachable from v
  - DFS will never be called on same vertex twice

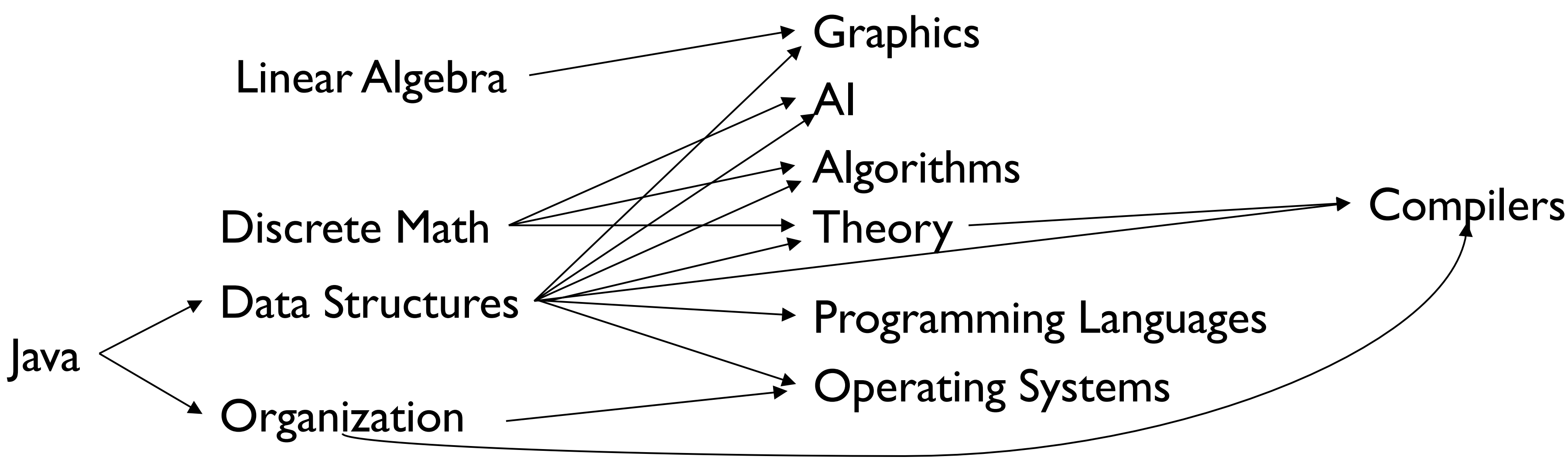
Recursive Depth-First Search

Claim: `DFS(G,v)` returns the number of unvisited nodes reachable from `v`

Proof: Uses previous two observations

- DFS visits every node reachable from `v`
- DFS doesn't visit any node *not* reachable from `v`

Directed Graphs

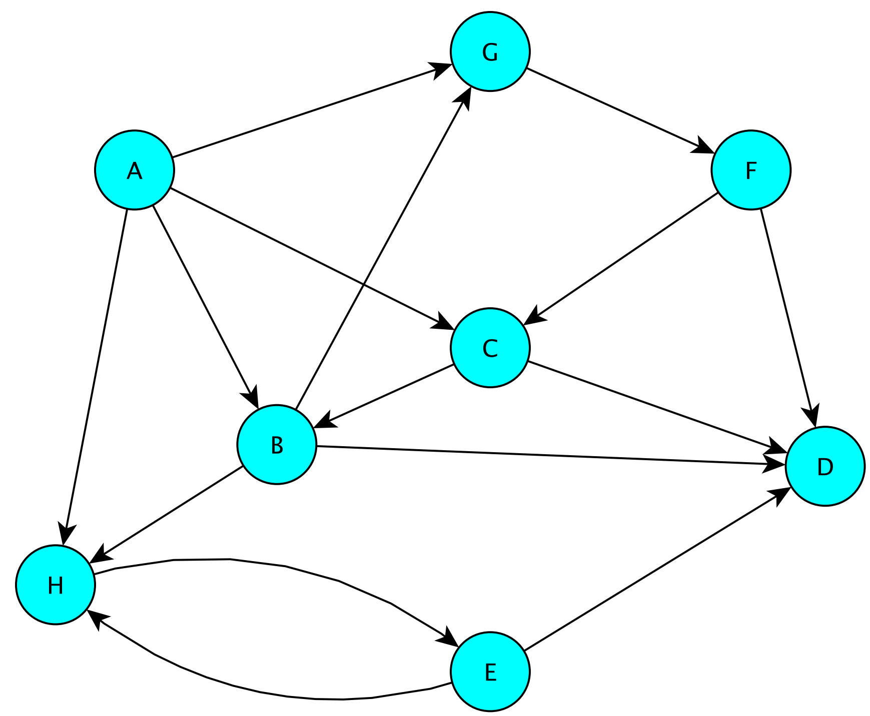


Definition: In a *directed graph* $G=(V,E)$, each edge e in E is an *ordered pair*: $e=(u,v)$ vertices: its *incident vertices*. The *source* of e is u; the *destination/target* is v.

Note: $(u,v) \neq (v,u)$

Directed Graphs

- The (out) *neighbors* of B are D, G, H: B has *out-degree* 3
- The *in neighbors* of B are A, C: B has *in-degree* 2
- A is a *source* in G: A has in-degree 0
- D is *sink* in G: D has out-degree 0



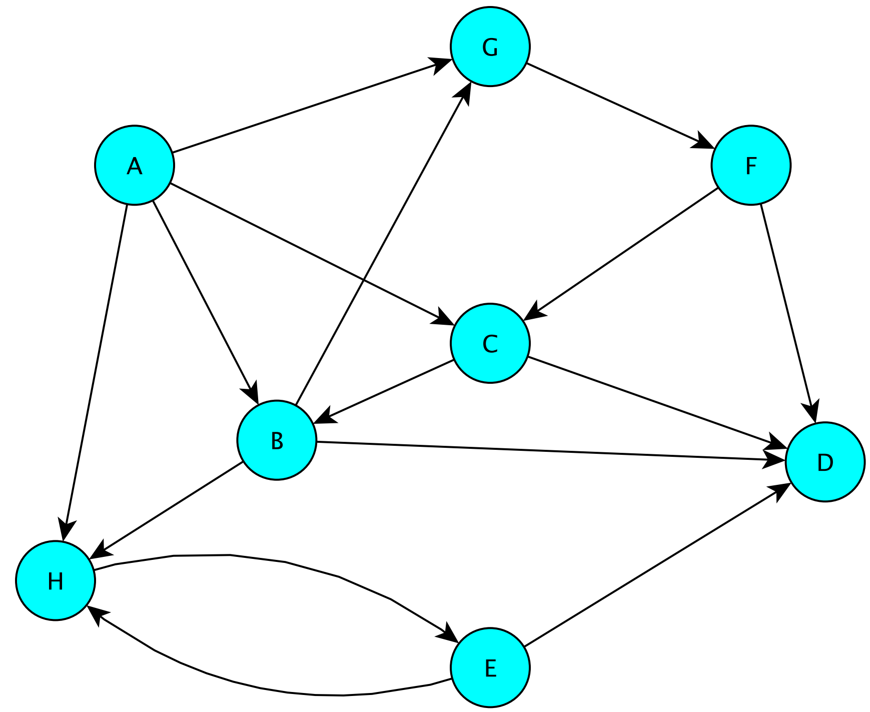
A walk is still an alternating sequence of vertices and edges

$$u = v_0, e_1, v_1, e_2, v_2, ..., v_{k-1}, e_k, v_k = v$$

but now $e_i = (v_{i-1}, v_i)$: all edges *point along direction* of walk

Directed Graphs

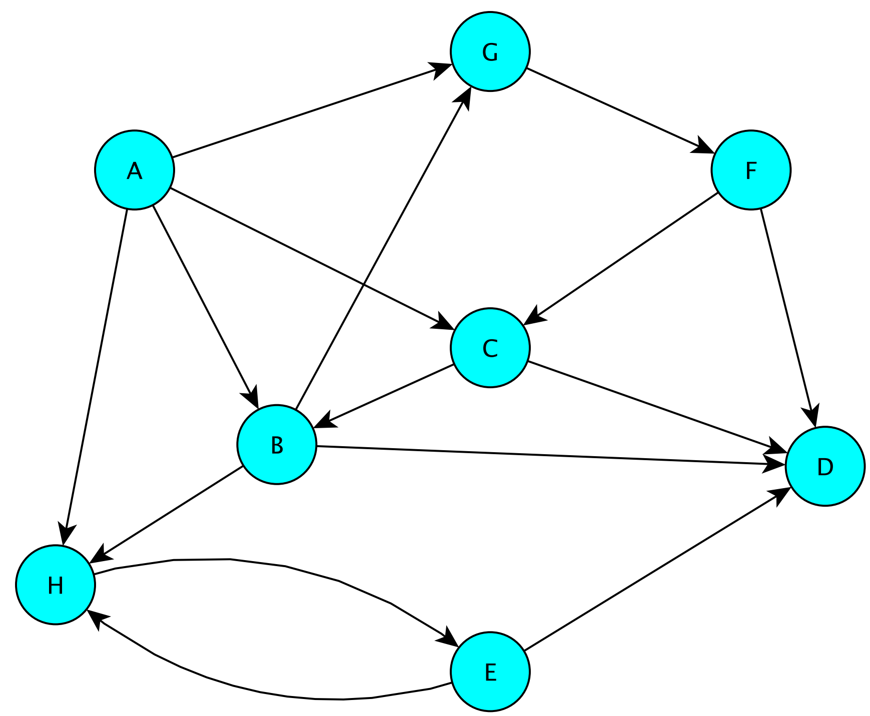
- A, B, H, E, D is a walk from A to D
- It's also a (simple) path
- D, E, H, B, A is *not* a walk from D to A
- B, G, F, C, B is a (directed) cycle (it's a 4-cycle)
- So is H, E, H (a 2-cycle)



- D is reachable from A (via path A, B, D), but A is not reachable from D
- In fact, every vertex is reachable from A

Directed Graphs

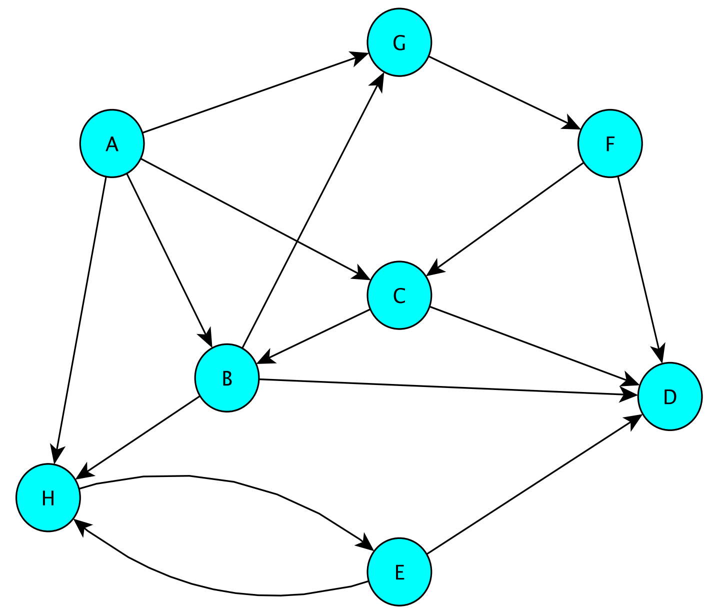
- A BFS of *G* from A visits every vertex
- A BFS of *G* from F visits all vertices but A
- A BFS of *G* from E visits only E, H, D

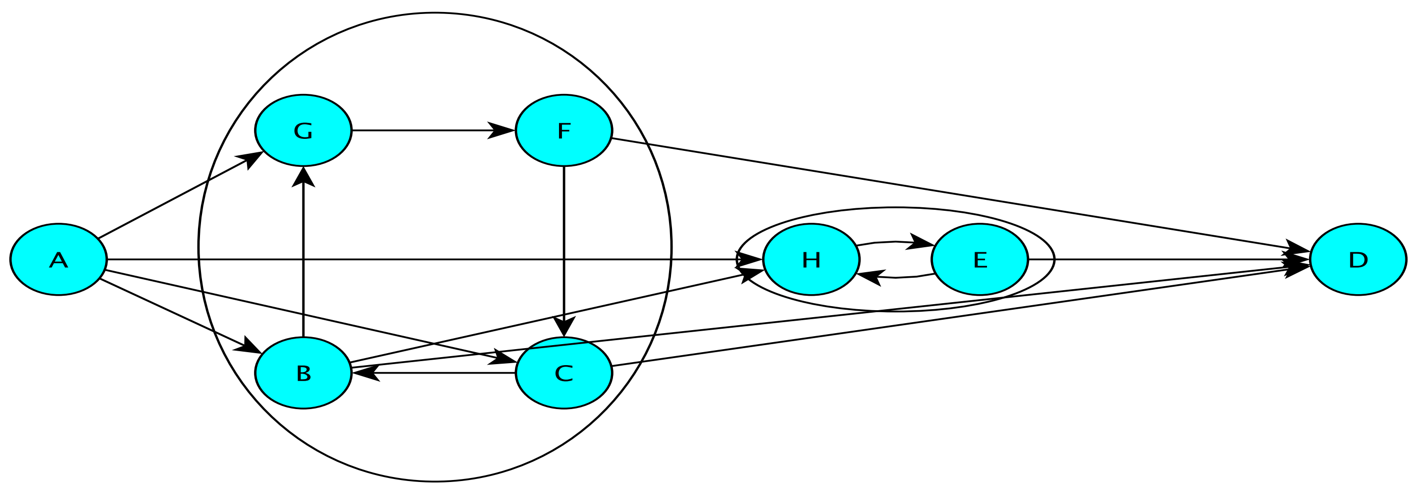


- Connectivity in directed graphs is more subtle than in undirected graphs!

Directed Graphs

- Vertices u and v are *mutually reachable* vertices if there are paths from u to v and v to u
- *Maximal* sets of mutually reachable vertices form the *strongly connected components* of G





Implementing Graphs

- Involves a number of implementation decisions, depending on intended uses
  - What kinds of graphs will be availabe?
    - Undirected, directed, mixed
  - What underlying data structures will be used?
  - What functionality will be provided
  - What aspects will be public/protected/private
- We'll focus on popular implementations for undirected and directed graphs (separately)