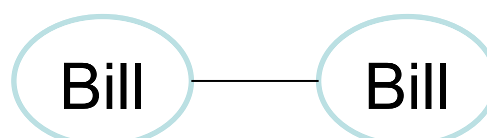
CSCI 136
Data Structures & Advanced Programming

Lecture 27

Fall 2017

Instructors: 

Last Time

- Introduction To Graphs
  - Definitions and Properties: Undirected Graphs

Today’s Outline

- More on Graphs
  - Applications and Problems
    - Testing connectedness
    - Counting connected components
      - Breadth-first and Depth-first search
  - Directed Graphs
    - Definition and Properties
  - Reachability and (Strong) Connectedness
- Graph Data Structures: Preliminaries
  - Graph Interface

Basic Definitions & Concepts

- **Definition:** An *undirected graph* `G = (V,E)` consists of two sets:
  - `V` : the *vertices* of `G`
  - `E` : the *edges* of `G`

- Each edge `e` in `E` is defined by a set of two vertices: its *incident vertices*
- We write `e={u,v}` and say that `u` and `v` are *adjacent*
- The *degree* of a vertex is the number of *incident edges* (loops counted twice)

Walking Along a Graph

- A *walk from u to v* in a graph $G = (V,E)$ is an *alternating* sequence of vertices and edges

$$u = v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k = v$$

such that each $e_i = \{v_i, v_{i+1}\}$ for $i = 1, \ldots, k$

- Note a walk starts and ends on a vertex
- If no *edge* appears more than once then the walk is called a *path*
- If no *vertex* appears more than once then the walk is a *simple path*

Walking In Circles

- A *closed walk* in a graph $G = (V,E)$ is a *walk*

  $$v_0, e_1, v_1, e_2, v_2, \dots, v_{k-1}, e_k, v_k$$

  such that $v_0 = v_k$ (it ends at the starting v)
- A *circuit* is a *path* where $v_0 = v_k$
  - Circuit vs. closed walk? Circuit has no repeat edges
- A *cycle* is a *simple* path where $v_0 = v_k$
  - Circuit vs. cycle? Cycle has no repeated vertices.
- The length of any of these is the number of *edges* in the sequence

Little Tiny Theorems

- If there is a walk from u to v, then there is a walk from v to u.
- If there is a walk from u to v, then there is a path from u to v (and from v to u)
- If there is a path from u to v, then there is a simple path from u to v (and v to u)
- Every circuit through v contains a cycle through v
- Not every closed walk through v contains a cycle through v! [Try to find an example!]

A Basic Graph Fact

- Denote the degree of a vertex *v* by *deg(v)*.
- Theorem: For any graph `G = (V,E)`

$$\sum_{v \in V} \deg(v) = 2|E|$$

where |`E`| is the number of edges in `G`

- Proof Hint: Induction on |`E`|: How does removing an edge change the equation?
  - Instead: Count pairs *(v,e)* where *v* is incident with e

Reachability and Connectedness

- **Definition:** A vertex v in G is *reachable* from a vertex u in G if there is a path from u to v
  - v is reachable from u *iff* u is reachable from v
- **Definition:** An undirected graph G is *connected* if for every pair of vertices (u, v) in G, v is reachable from u (and vice versa)
- The set of all vertices reachable from v, along with all edges of G connecting any two of them, is called the *connected component* of v

Basic Graph Algorithms

- We'll look at a number of graph algorithms
  - Connectedness: Is G connected?
    - If not, how many *connected components* does G have?
  - Cycle testing: Does G contain a cycle?
    - Does G contain a cycle through a given vertex?
  - If the edges of G have costs:
    - What is the cheapest subgraph connecting all vertices
      - Called a *connected, spanning subgraph*
    - What is a cheapest path from u to v?
  - And more....

Operations on Graphs

- What are the basic operations we need to describe algorithms on graphs?
  - Given vertices u and v: are they *adjacent*?
  - Given vertex v and edge e, are they *incident*?
  - Given an edge e, get its incident vertices (*ends*)
  - How many vertices are adjacent to v? (*degree* of v)
    - The vertices adjacent to v are called its *neighbors*
  - Get a list of the vertices *adjacent* to v
    - From which we can get the edges *incident* with v

Testing Connectedness

- How can we determine whether G is connected?
  - Pick a vertex v; see if every vertex u is reachable from v
- How could we do this?
  - Visit the neighbors of v, then visit their neighbors, etc. See if you reach all vertices
    - Assume we can mark a vertex as "visited"
- How do we manage all of this visiting?
  - Let's try an example…

Reachability: Breadth-First Search

BFS(G, v)   *// Do a breadth-first search of G starting at v*

// pre: all vertices are marked as unvisited

count $\leftarrow$ *0;*

Create empty queue Q; enqueue v; mark v as visited; count++

While Q isn't empty

  current $\leftarrow$ *Q.dequeue();*

  for each unvisited neighbor u of current :

    add u to Q; mark u as visited; count++

return count;

Now compare value returned from `BFS(G,v)` to |`V`|