CSCI 136
Data Structures &
Advanced Programming

Lecture 25

Fall 2017

Instructors: Bill^2

Last Time

- Binary search trees (Ch 14)
  - Implement the `OrderStructure` interface
  - The `locate(root, value)` method
    - returns either: node that stores value, or parent where value would be inserted
  - Many methods use `locate`
    - `contains`
    - `add`
    - `remove`
    - `...`

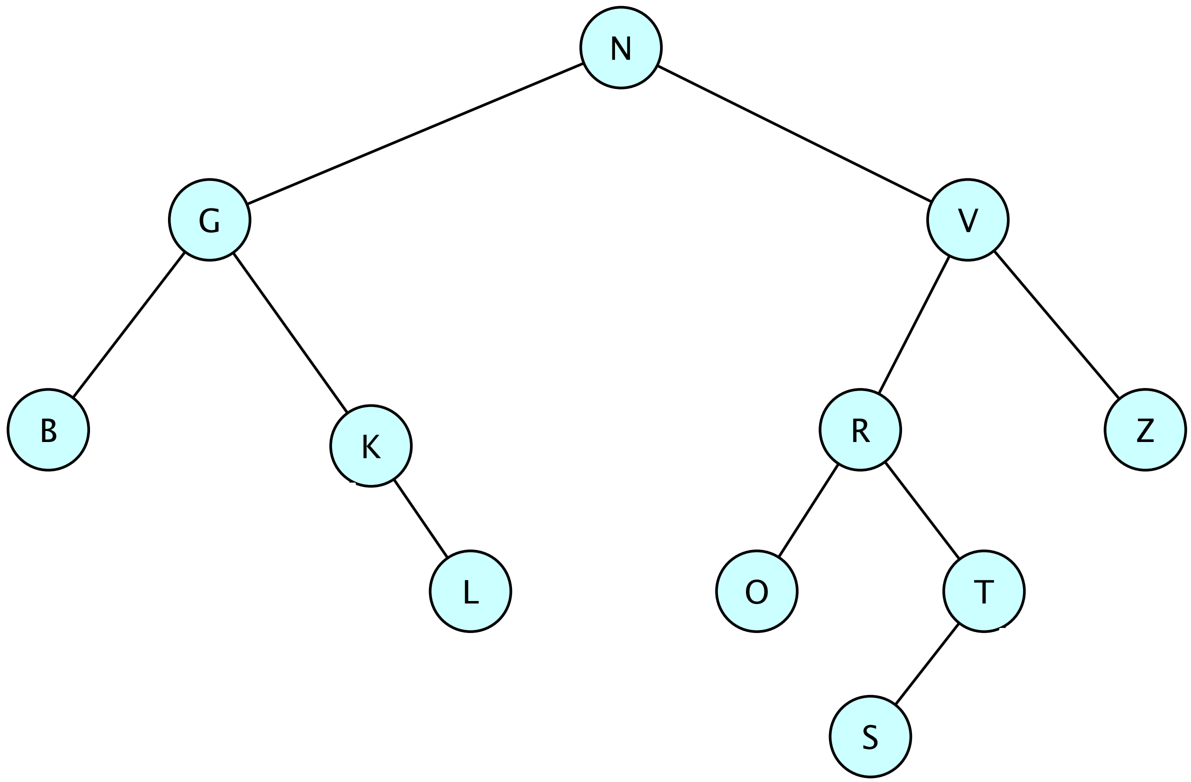
Today's Outline

- Binary search trees (Ch 14)
  - Finish OrderStructure API
    - `add()/remove()`
  - Tree balancing to maintain small height
    - `rotate()`
  - Partial taxonomy of balanced tree species
    - AVL Trees
    - Splay Trees
    - Red-Black Trees

Binary Search Trees

- A binary tree is a binary *search* tree if it is:
  - Empty, or
  - All nodes in the left subtree are less than or equal to the root, all nodes in the right subtree are greater than or equal to the root, and the left and right subtrees are binary search trees.
- In our implementation, right subtrees only hold values that are *strictly greater than* the root
  - Why?

Add: Duplicate Values



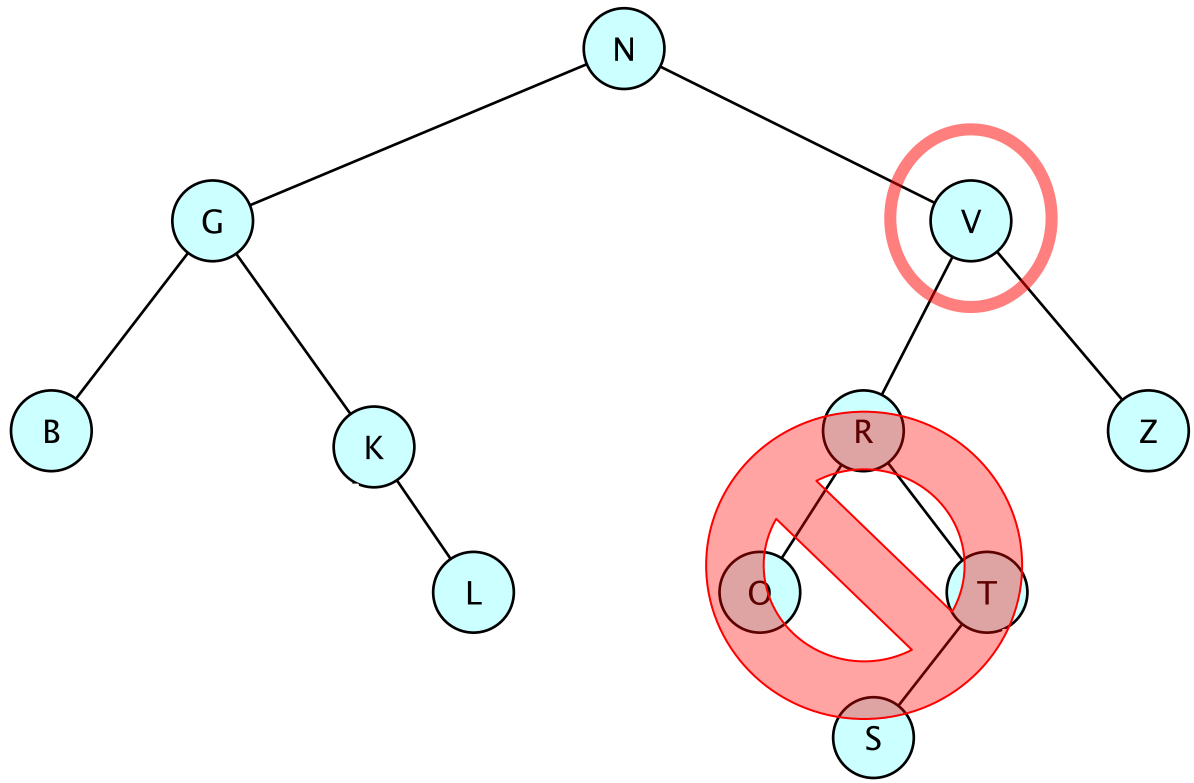
Where would a new K be added?
A new V?

First (Bad) Attempt: add(E value)

```
public void add(E value) {
    BinaryTree<E> newNode = new BinaryTree<E>(value,EMPTY,EMPTY);
    if (root.isEmpty()) {
        root = newNode;
    } else {
        BinaryTree<E> insertLocation = locate(root,value);
        E nodeValue = insertLocation.value();
        if (ordering.compare(value,nodeValue) > 0)
            insertLocation.setRight(newNode); // value > nodeValue
        else
            insertLocation.setLeft(newNode); // value <= nodeValue
    }
    count++;
}
```

Problem: If duplicate values are allowed in the BST, the left subtree might not be empty when setLeft is called

How to Add Duplicate Values



How to perform: bst.add("v") ???

locate("v").setLeft(new BinaryTree ("v")); ???

Strategy: Add Duplicates to Predecessor

- If `insertLocation` has a left child:
  - Find `insertLocation`'s *predecessor*, then
  - Add duplicate node as *right child* of predecessor
    - Why?
    - Relationships among `root`, `pred(root)`, and `node`?

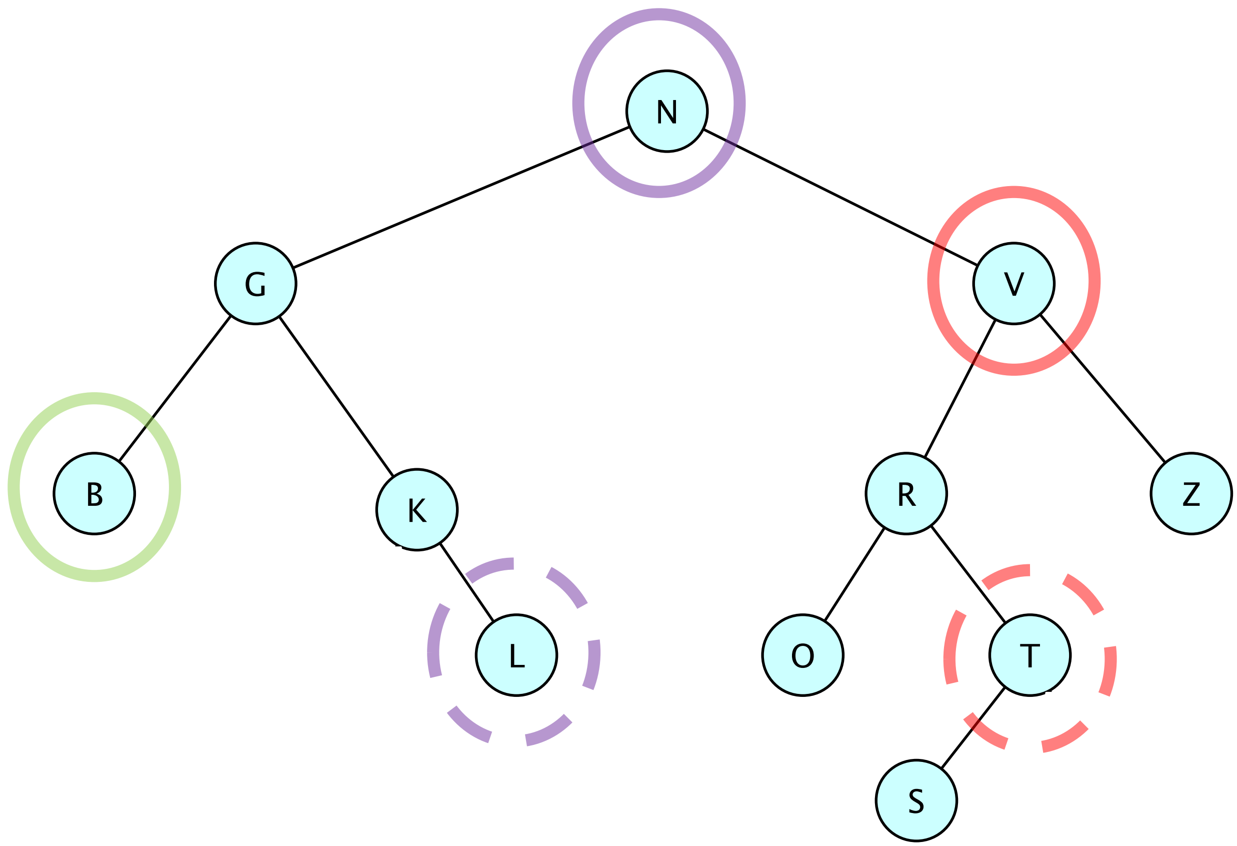
  - Make duplicate values the successor of their predecessor!

Corrected: add(E value)

```
public void add(E value) {
    BinaryTree<E> newNode = new BinaryTree<E>(value,EMPTY,EMPTY);
    if (root.isEmpty()) {
        root = newNode;
    } else {
        BinaryTree<E> insertLocation = locate(root,value);
        E nodeValue = insertLocation.value();
        if (ordering.compare(value,nodeValue) > 0) {
            // value > nodeValue
            insertLocation.setRight(newNode);
        } else {
            // value <= nodeValue
            if (insertLocation.left().isEmpty())
               insertLocation.setLeft(newNode);
            else
               predecessor(insertLocation).setRight(newNode);
    }
    count++;
}
```

Recap: If value is in the tree, insert newNode immediately before it (successor of predecessor)

How to Find Predecessor?



predecessor(root) is the rightmost node in root's left subtree

Predecessor

```
// return node with largest value in root's left subtree
// pre: root is not empty, root's left child is not empty
protected BinaryTree<E> predecessor(BinaryTree<E> root) {
    BinaryTree<E> result = root.left();

    while (!result.right().isEmpty())
        result = result.right();

    return result;
}
```

"slide down" the left subtree

Removal

- If we can remove the root, we can remove any element in a BST in the same way
  - Why?
- We need to implement:
  - `public E remove(E item)`
- We can benefit from a helper:
  - `protected BT removeTop(BT top)`
    - `removeTop(BT top)` removes `top`, and returns the root node of the resulting tree
- Assuming `removeTop` works, let's implemet remove

BST remove()

```
public E remove(E value) {
    // base case 1: empty tree
    if (isEmpty()) return null;

    // base case 2: root contains value
    if (value.equals(root.value())) {
        E result = root.value();
        count--;
        root = removeTop(root);
        return result;
    }

    . . .
```

BST remove()

```
    // general case: find node that holds value, remove node,
    //      and re-attach resulting tree at node's old location
    BinaryTree<E> location = locate(root,value);
    if (value.equals(location.value())) {  // found node with value
        count--;
        BinaryTree<E> parent = location.parent();
        if (parent.right() == location) { // removing right child
            parent.setRight(removeTop(location));
        } else {                          // removing left child
            parent.setLeft(removeTop(location));
        }
        return location.value();
    }

    // value not found in tree, nothing to do
    return null;
}
```

RemoveTop(topNode)

Detach left and right sub-trees from root (i.e. topNode)

If either left or right is empty, ***return*** *the other one* Cases 1 & 2

If left has no right child Case 3

make right the right child of left then ***return*** *left*

Otherwise find largest node C in left General case

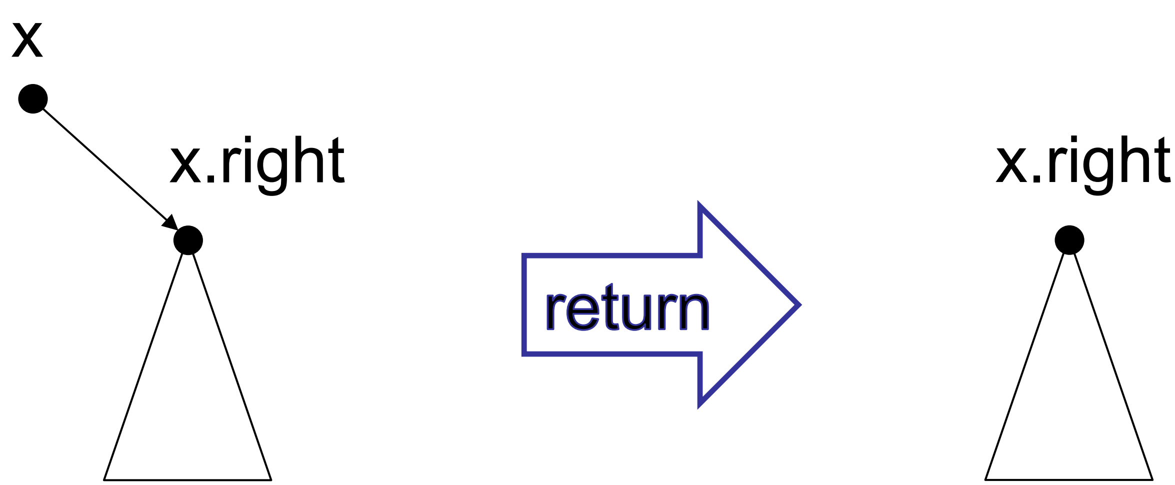
// C is the right child of its own parent P

// C is the predecessor of right (ignoring topNode)

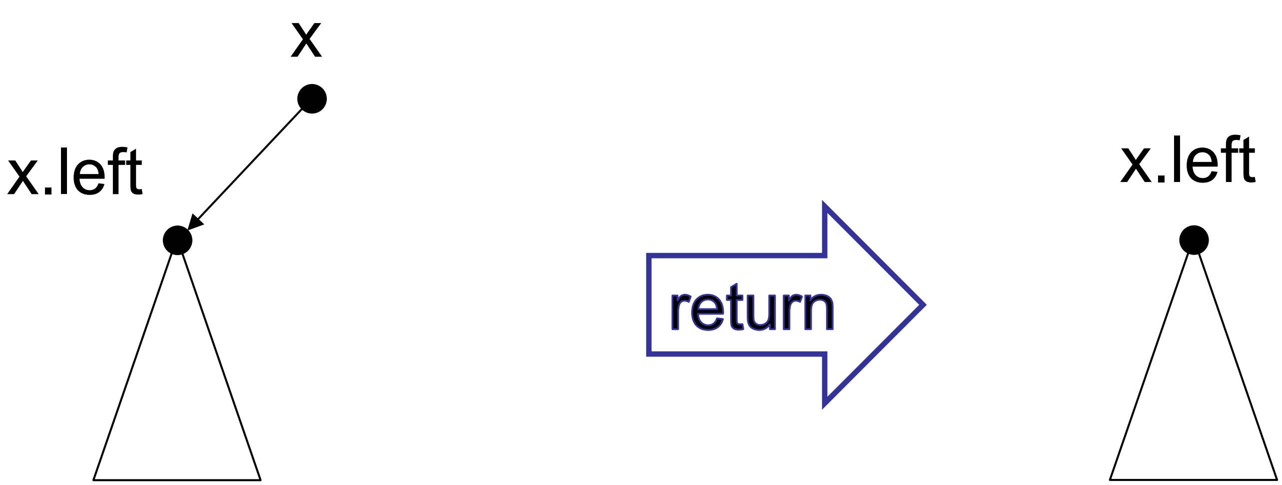
Detach C from P; make C's left child the right child of P

Make C new root with left and right as its sub-trees

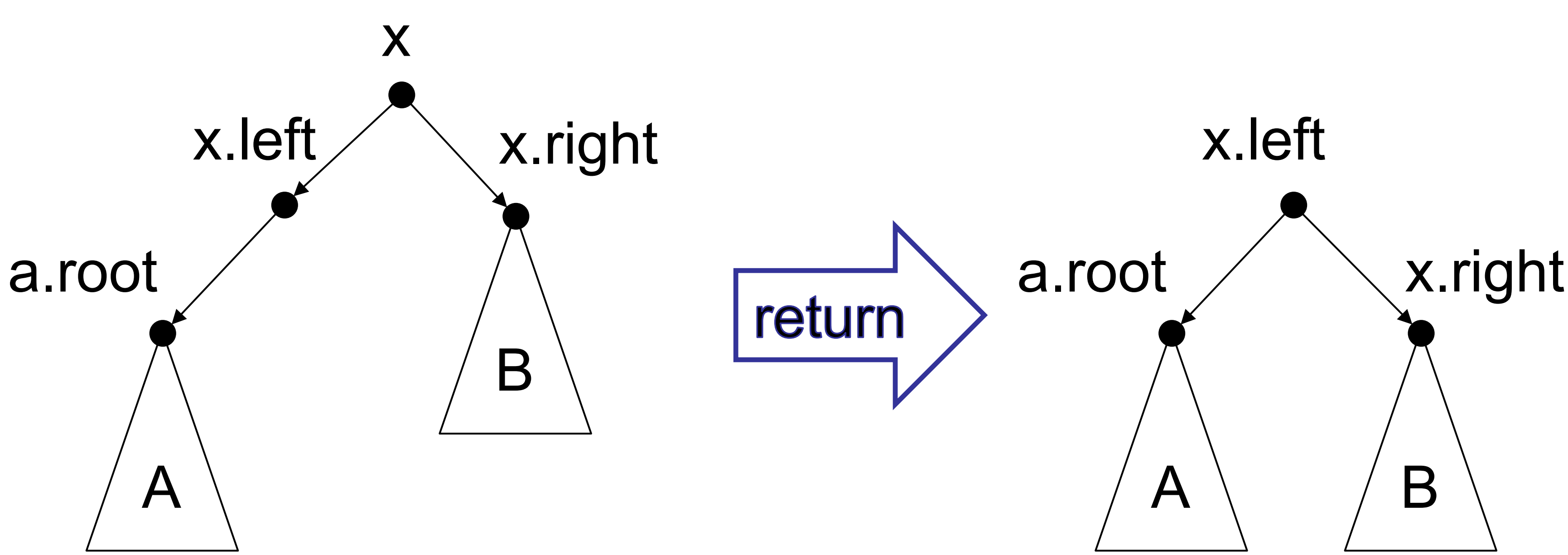
Case 1: No left binary tree



Case 2: No right binary tree



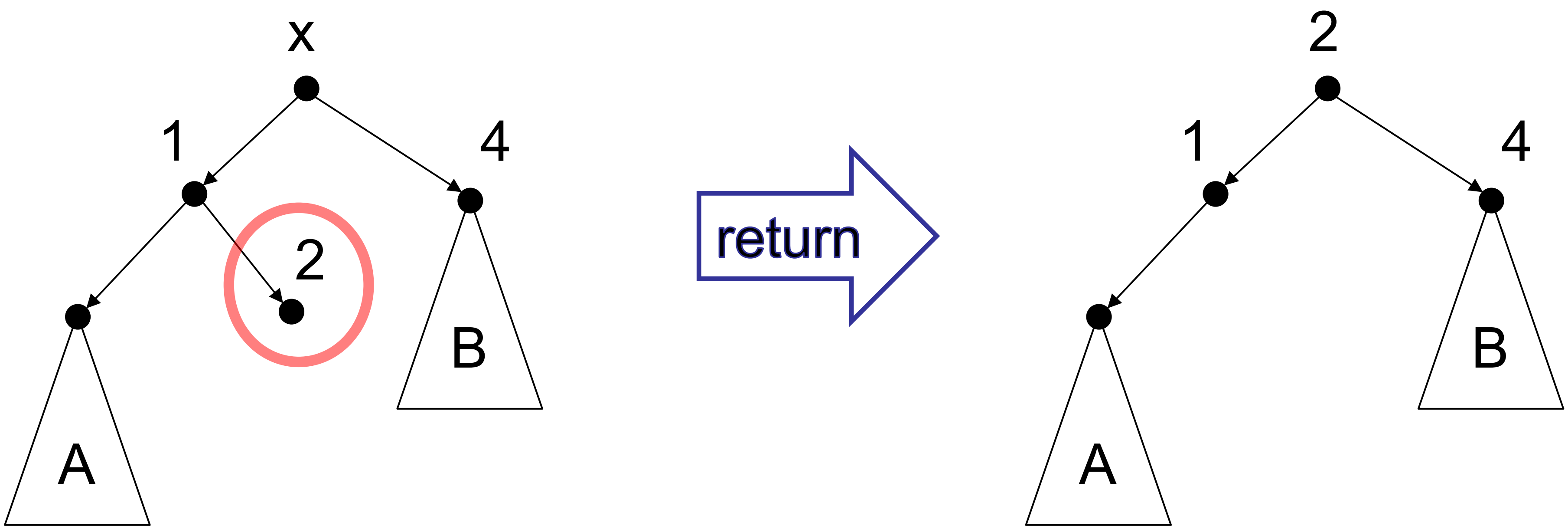
Case 3: Left has no right subtree



Case 4: General Case (HARD!)

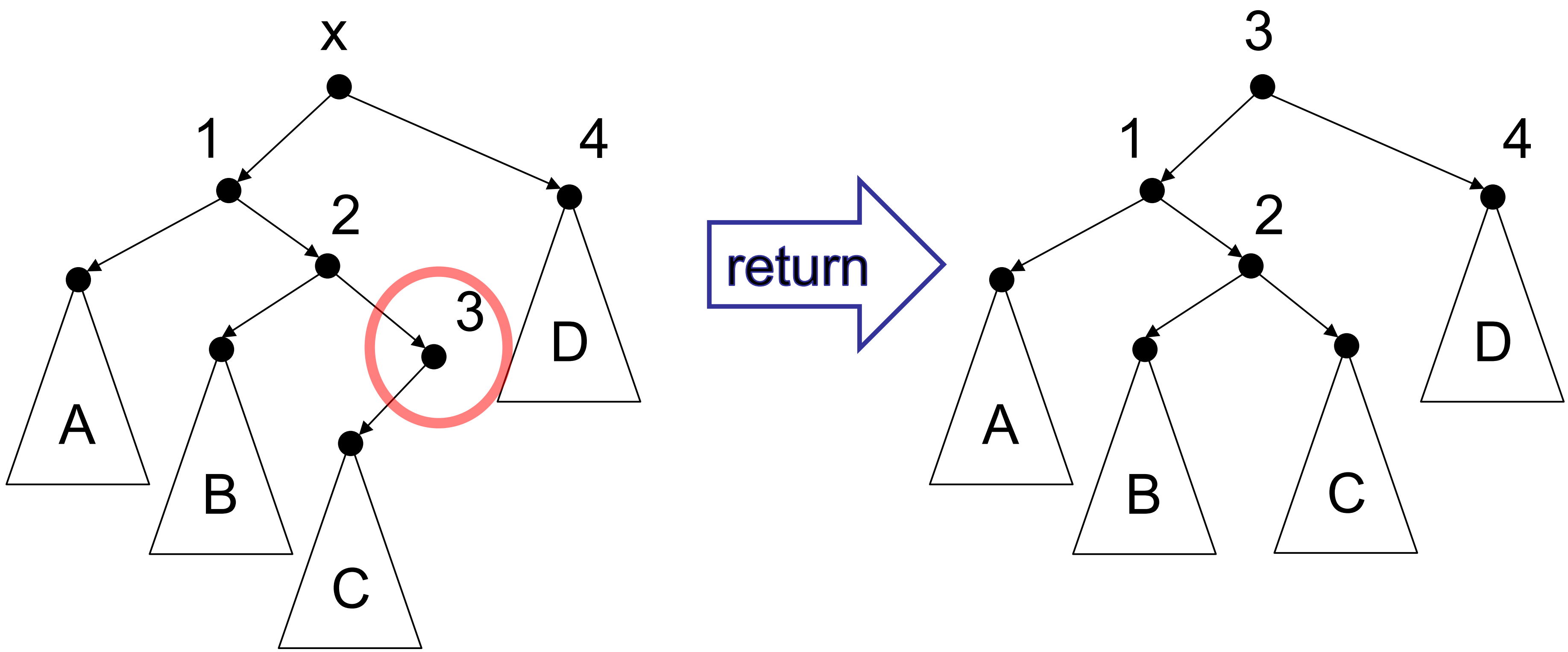
- Consider BST requirements:
  - Left subtree must be <= root
  - Right subtree must be >= root
- Strategy: replace the root with the largest value that is less than or equal to it
  - predecessor(root) : rightmost left descendant
- This may require reattaching the predecessor's left subtree!

Case 4: General Case (HARD!)



Replace root with predecessor(root),
then patch up the remaining tree

Case 4: General Case (HARD!)



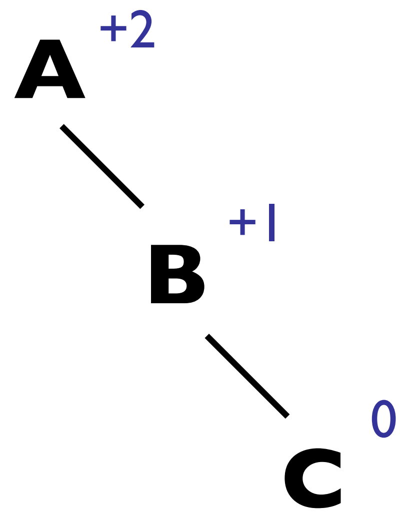
Replace root with predecessor(root), then patch up the remaining tree

Let's Write Some Code

- BinarySearchTree.java

But What About Height?

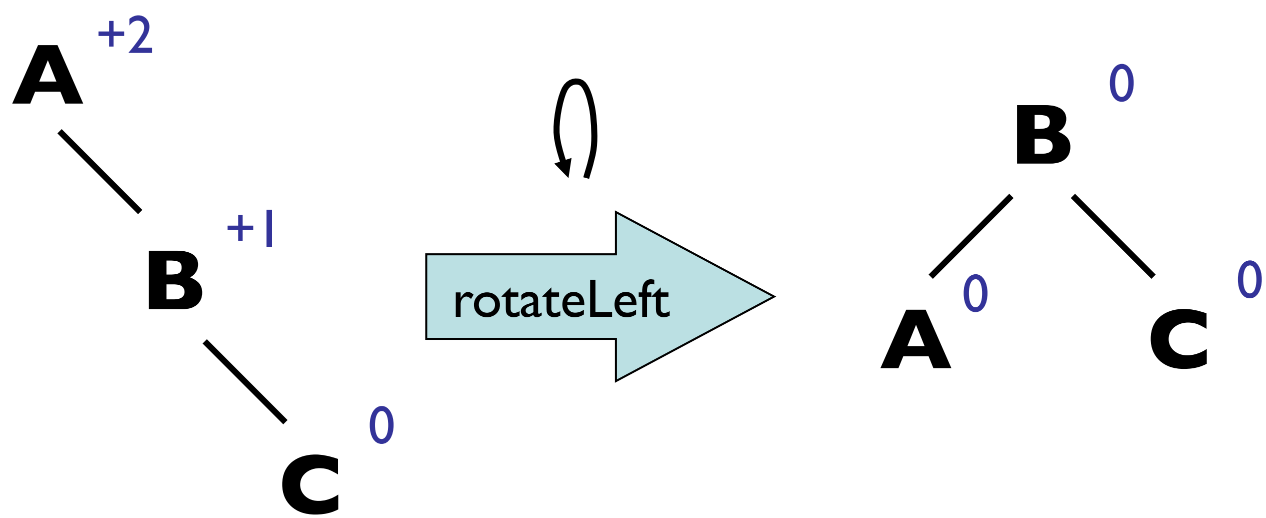
- Operations' performance all depend on *h*
- Can we design a binary search tree that is always "shallow" (minimizes h)?
- Yes! In many ways.
- AVL trees are one example
  - Named after its two inventors, G.M. Adelson-Velsky and E.M. Landis, who published a paper about AVL trees in 1962 called "An algorithm for the organization of information"



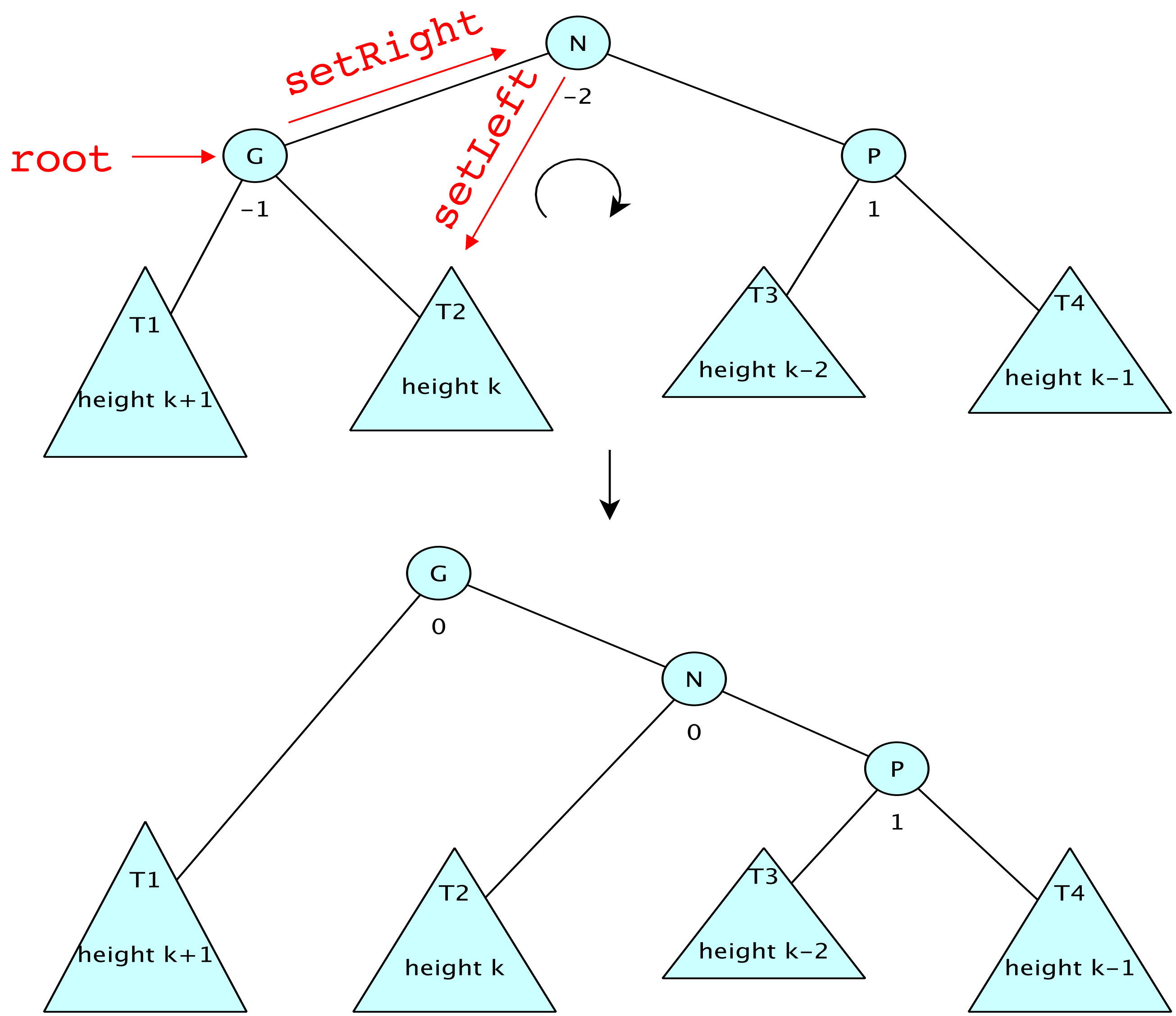
- The *balance factor* of a node is the height of its right subtree minus the height of its left subtree.
- A node with balance factor 1, 0, or -1 is considered *balanced*.
- A node with any other balance factor is considered *unbalanced* and requires rebalancing the tree.

Single Rotation

Unbalanced trees can be rotated to achieve balance.



Single Right Rotation

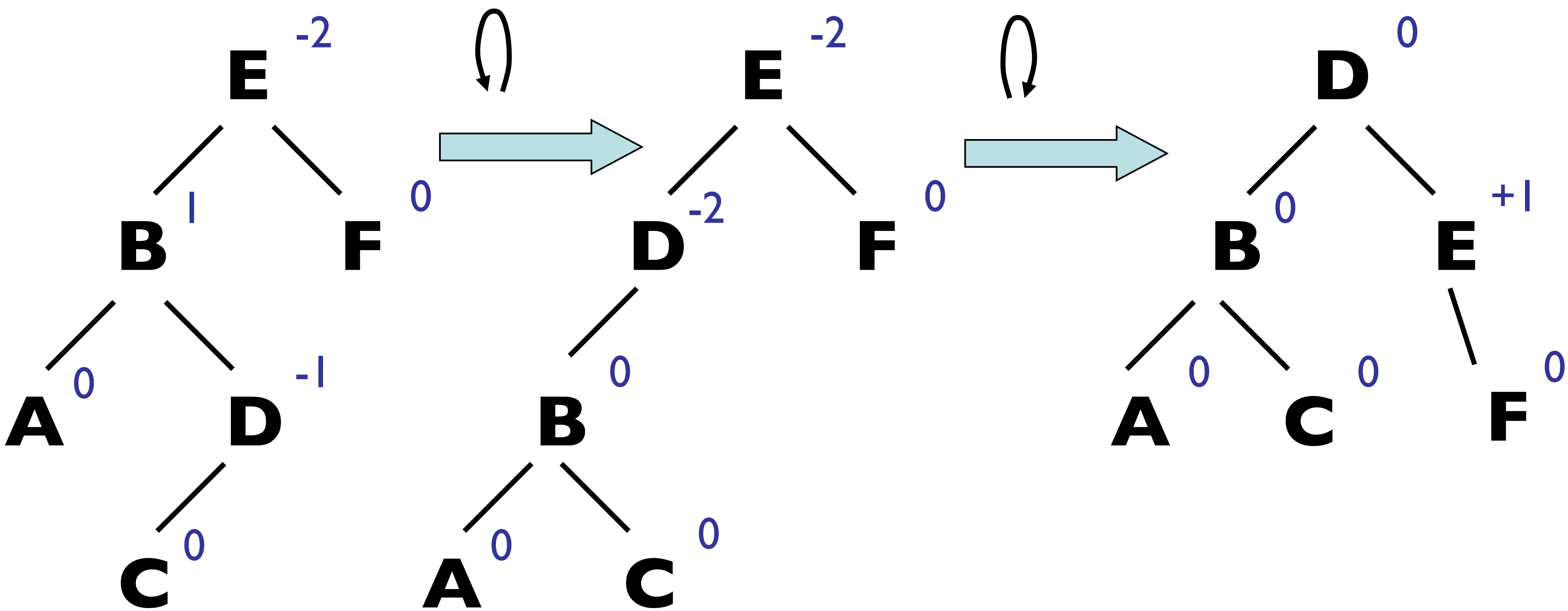


BinaryTree rotateRight()

```
// pre: this has a left subtree
// post: rotates local portion of tree so left child is root
protected void rotateRight() {
    // establish pointers/relationships before mucking with the tree
    BinaryTree<E> parent = parent;
    BinaryTree<E> newRoot = left();
    boolean wasChild = parent != null;
    boolean wasLeftChild = isLeftChild();

    // rotate!
    setLeft(newRoot.right());  // hook in new root
    newRoot.setRight(this); // make old root right child of new root
    if (wasChild) {
        // update parent pointers to rotated subtree
        if (wasLeftChild) parent.setLeft(newRoot);
        else  parent.setRight(newRoot);
    }
}
```

Double Rotation



AVL Tree Facts

- A tree that is AVL except at root, where root balance factor equals ± 2 can be rebalanced with at most 2 rotations
- add(v) requires at most O(log n) balance factor changes and one (single or double) rotation to restore AVL structure
- remove(v) requires at most O(log n) balance factor changes and O(log n) (single or double) rotations to restore AVL structure
- An AVL tree on n nodes has height O(log n)

AVL Trees have O(log n) Height

An AVL tree on n nodes has height O(log n)

Proof idea

- Show that an AVL tree of height h has at least fib(h) nodes (easy induction proof---try it!)
- Recall (HW): $fib(h) \geq \left({}^{3}/_{2}\right)^{h}$ if h ≥ 10
- So $n \geq \left({}^{3}/_{2}\right)^{h}$ and thus $\log_{3/2} n \geq h$
  - Recall that for any $a, b > 0$, $\log_a n = \frac{\log_b n}{\log_b a}$
  - So $\log_a n$ and $\log_b n$ are Big-O of one another
- So h is O(log n)

AVL Trees: One of Many

- There are many strategies for tree balancing to preserve O(log n) height, including
- AVL Trees: guaranteed O(log n) height
- Red-black trees: guaranteed O(log n) height
- B-trees (not binary): guaranteed O(log n) height
  - 2-3 trees, 2-3-4 trees, red-black 2-3-4 trees, ...
- Splay trees: *Amortized* O(log n) time operations
- Randomized trees: O(log n) *expected* height

Splay Trees

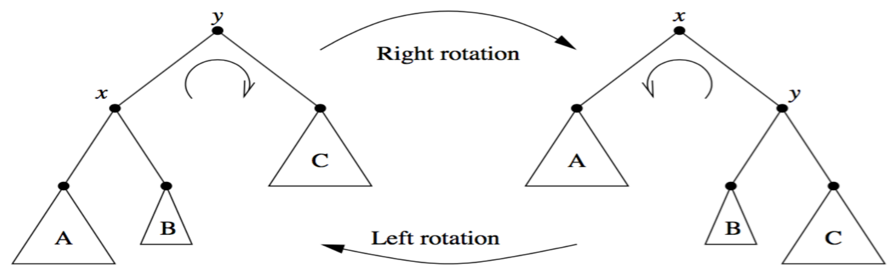
Splay trees are self-adjusting binary trees

- Each time a node is accessed, it is moved to root position via rotations
- No guarantee of balance (or shallow height)
- But good *amortized* performance

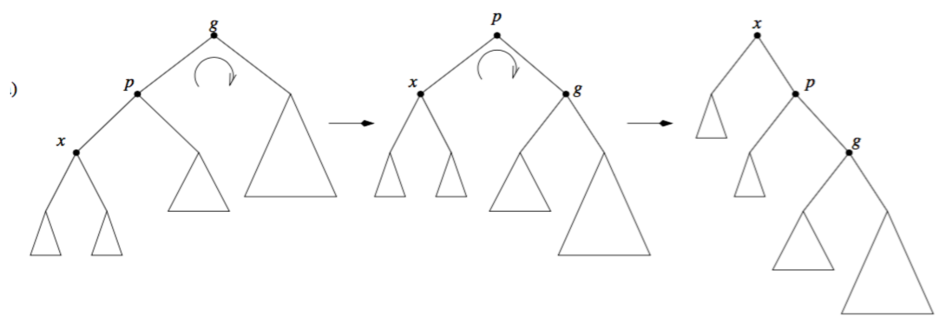
Theorem: Any set of m operations (add, remove, contains, get) on an n-node splay tree take at most O(m log n) time.

Splay Tree Rotations

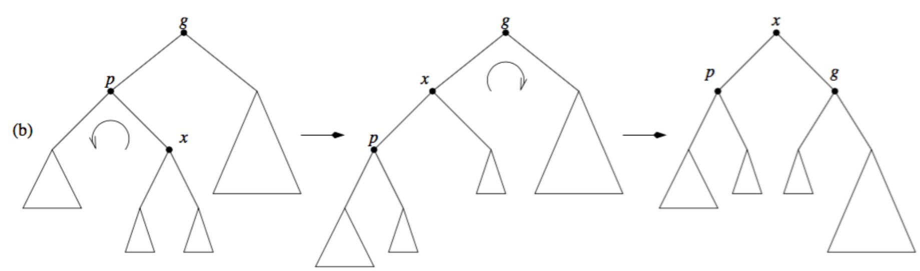
Right Zig Rotation (left version too)



Right Zig-Zig Rotation (left version too)



Right Zig-Zag Rotation (left version too)



Splay Tree Iterator

- Even *contains* method changes splay tree shape
- This breaks the standard in-order iterator!
  - Because the stack is based on the shape of the tree
- Solution: Remove the stack from the iterator
- Observation: Given location of current node (node whose value is next to be returned), we can compute it's (in-order)successor in *next()*
  - It's either left-most leaf of right child of current, or
  - It's closest "left-ancestor" of current
    - Ancestor whose left child is also an ancestor of current
- Also, reset must "re-find" root
  - Idea: Hold a single "reference" node, use it to find root

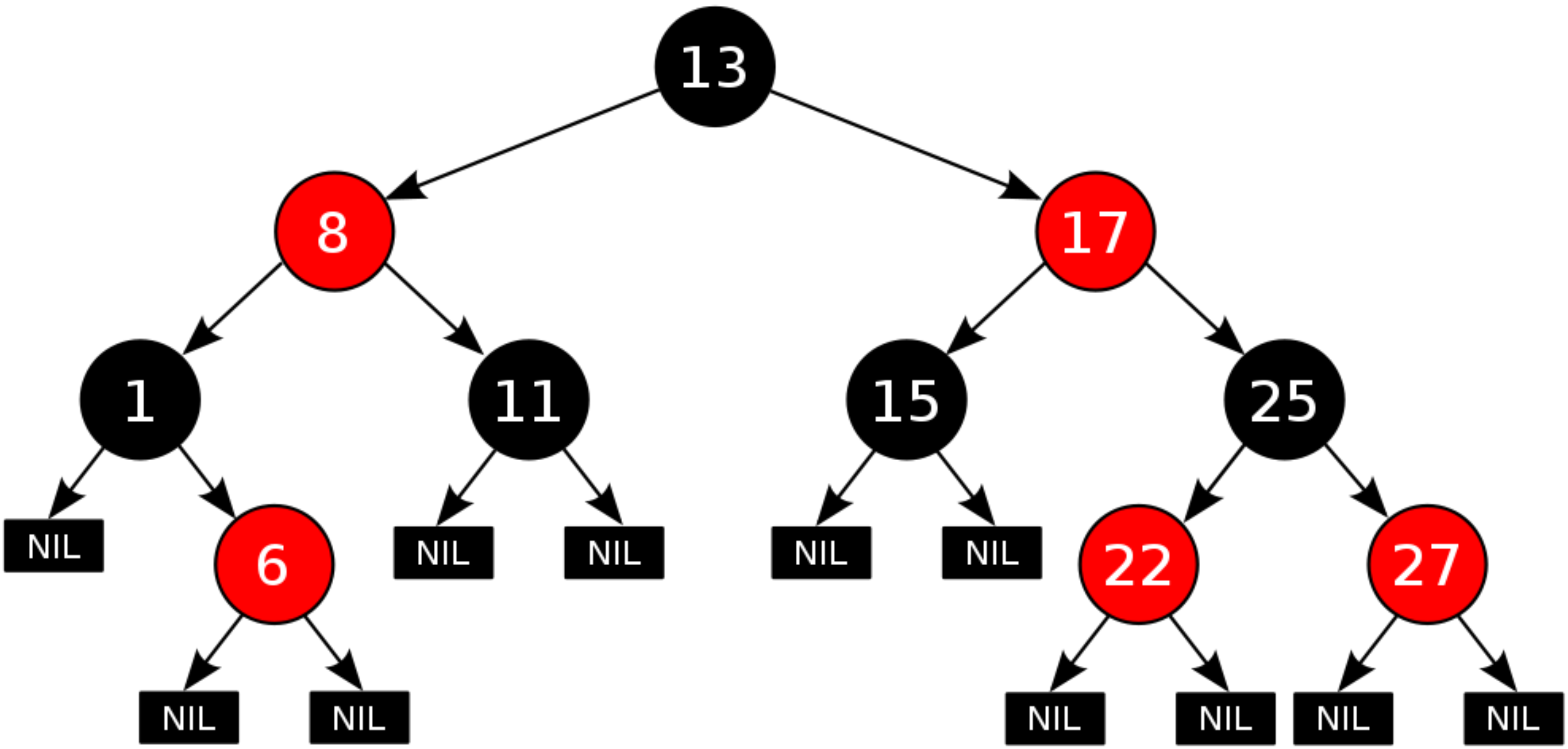
Red-Black Trees

Red-Black trees, like AVL, guarantee shallowness

- Each node is colored *red* or *black*
- Coloring satisfies these rules
  - All empty trees are black
    - We consider them to be the leaves of the tree
  - Children of red nodes are black
  - All paths from a given node to it's descendent leaves have the *same number* of black nodes
    - This is called the *black height* of the tree

A Red-Black Tree

(from Wikipedia.org)



Red-Black Trees

The coloring rules lead to the following result

Proposition: No leaf has depth more than twice that of any other leaf.

This in turn can be used to show

Theorem: A Red-Black tree with n internal nodes has height satisfying $h \leq 2\log(n+1)$

Note: The tree will have *exactly* n+1 (empty) leaves

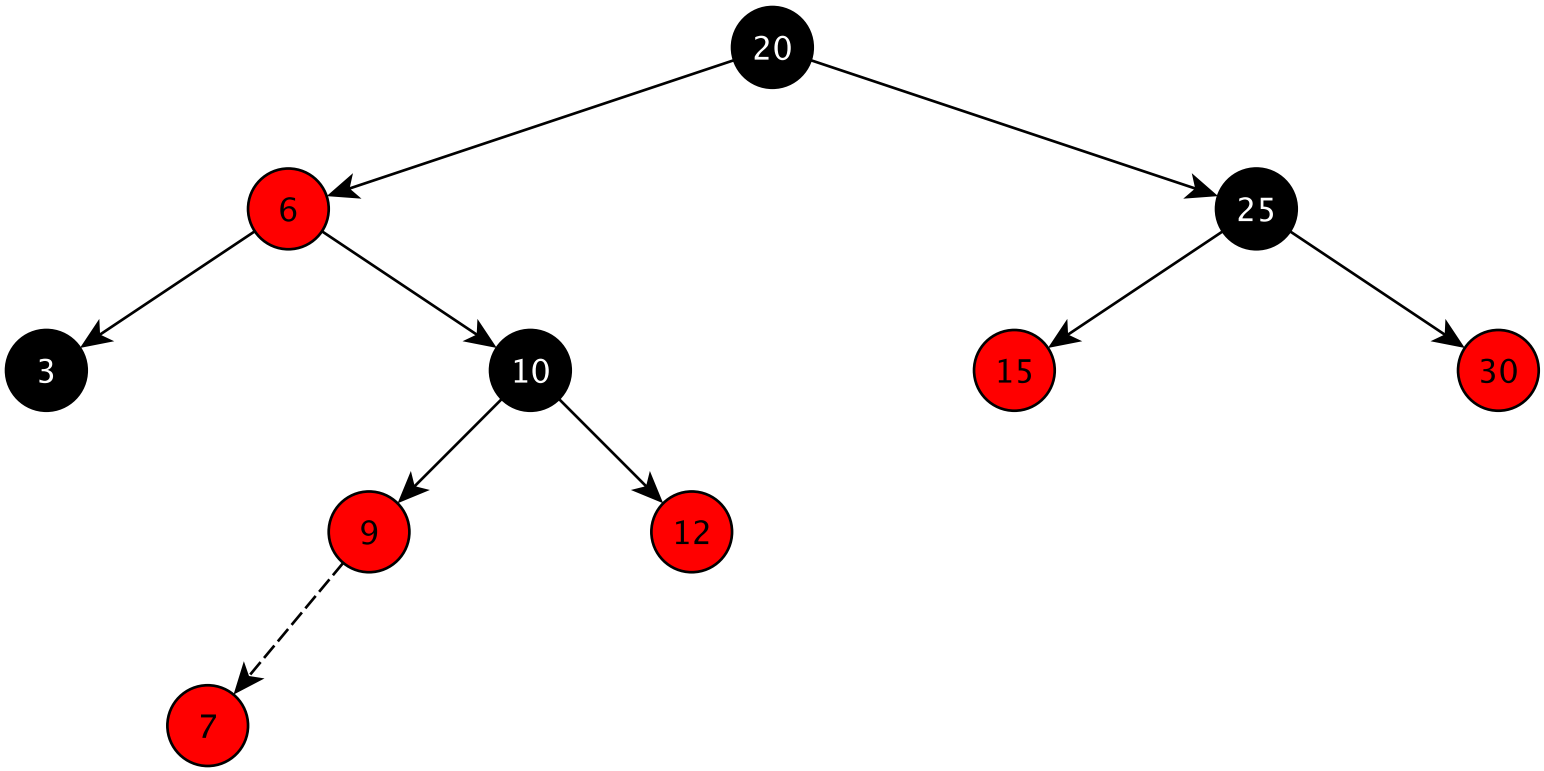
Red-Black Trees

Theorem: A Red-Black tree with n *internal* nodes has height satisfying $h \leq 2\log(n+1)$

Proof sketch: Note: we count empty tree nodes!

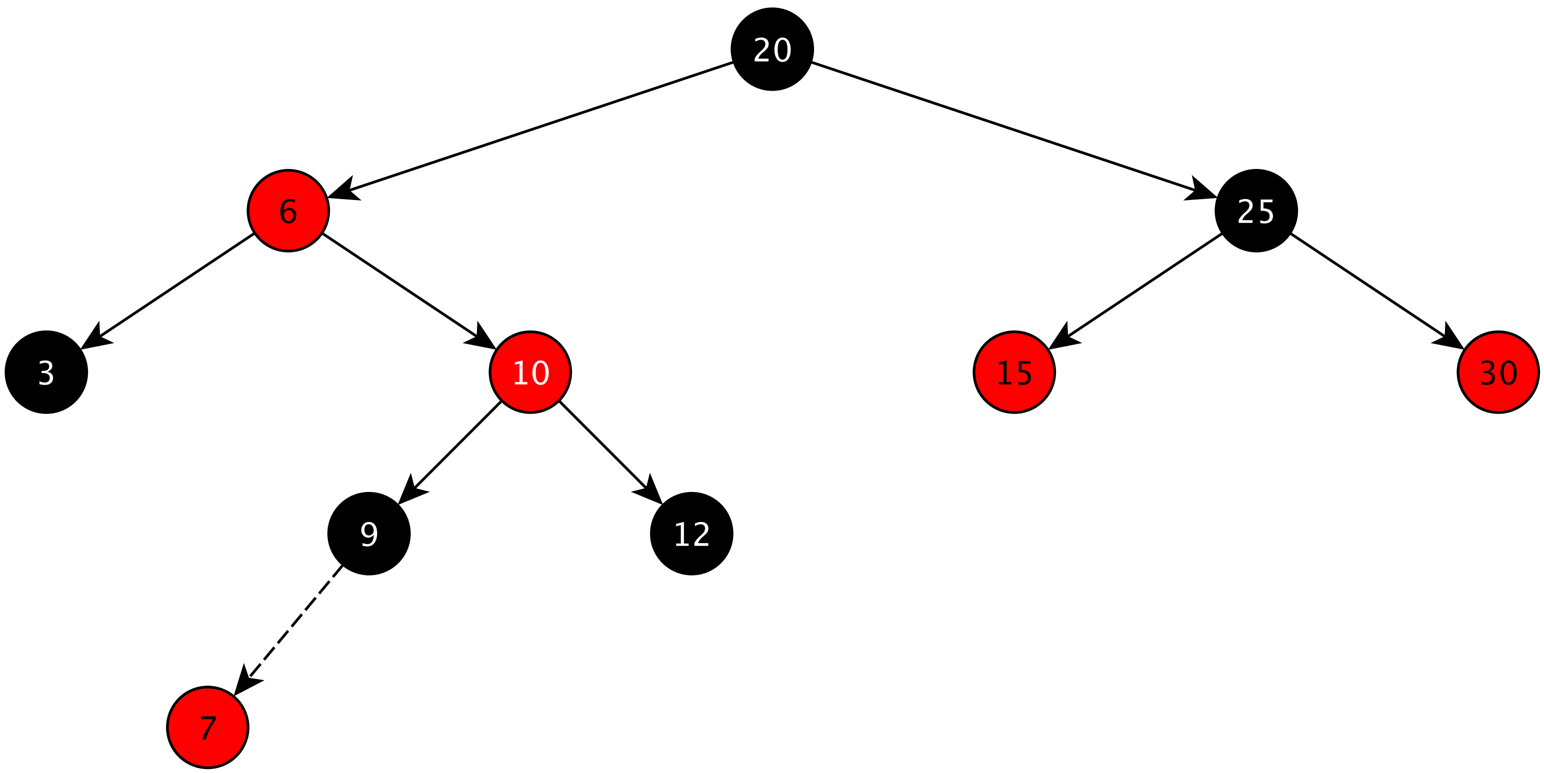
- If root is red, recolor it black.
- Now merge red children into (black) parents
  - Now n' ≤ n nodes and height h' ≥ h/2
- New tree has all children with degree 2, 3, or 4
  - All leaves have depth *exactly* h' and there are n+1 leaves
    - So $n+1 \geq 2^{h'}$, so $\log_2(n+1) \geq h' \geq \frac{h}{2}$
- Thus $2\log_2(n+1) \geq h$

Red-Black Tree Insertion



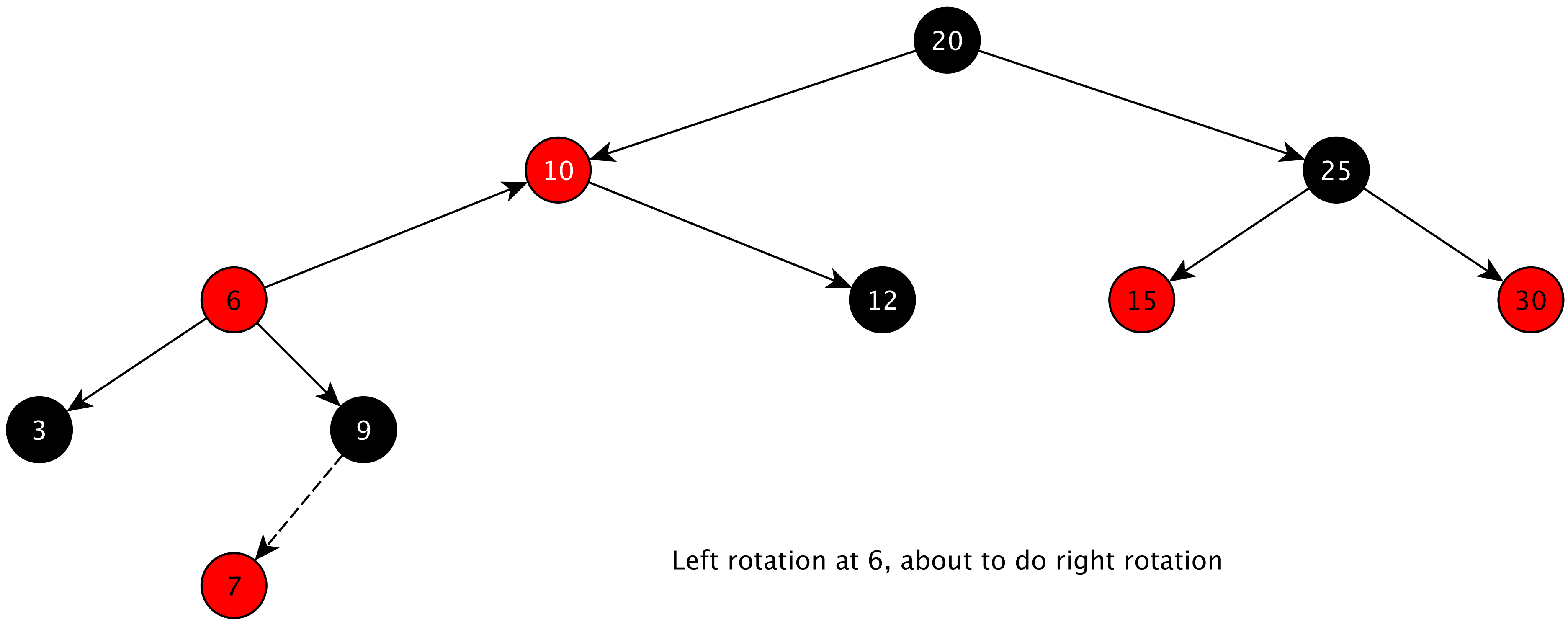
Black empty leaves not drawn. 7 just added Black-height still 2.

Red-Black Tree Insertion



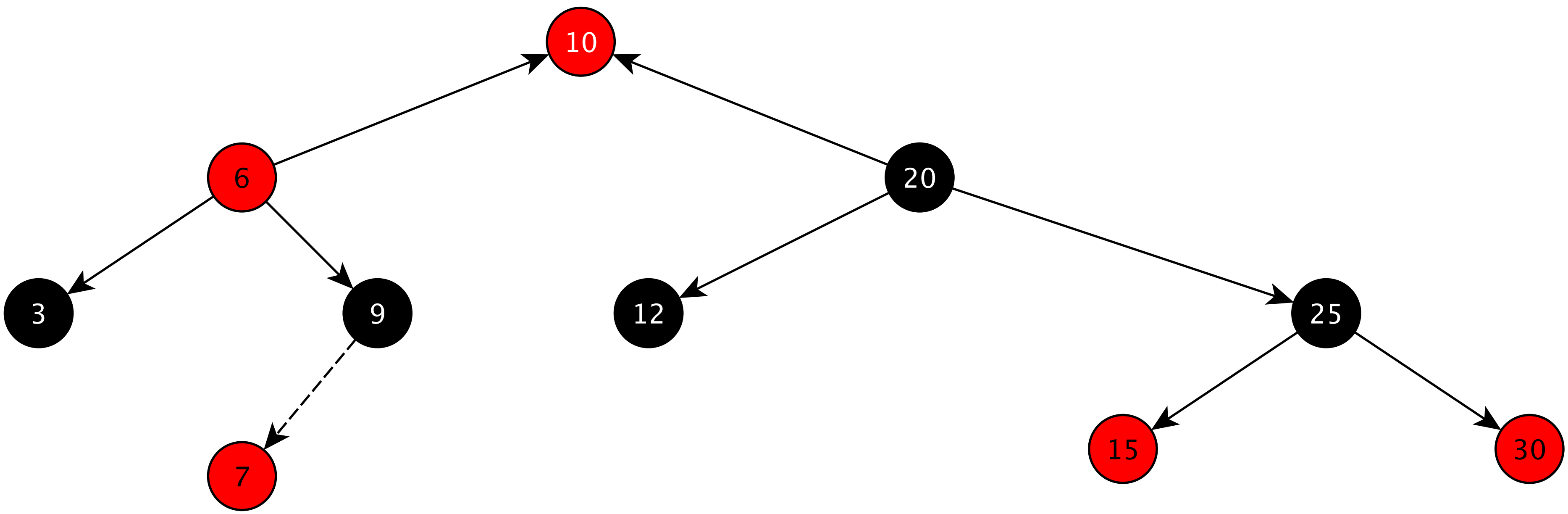
Black height still 2, color violation moved up

Red-Black Tree Insertion



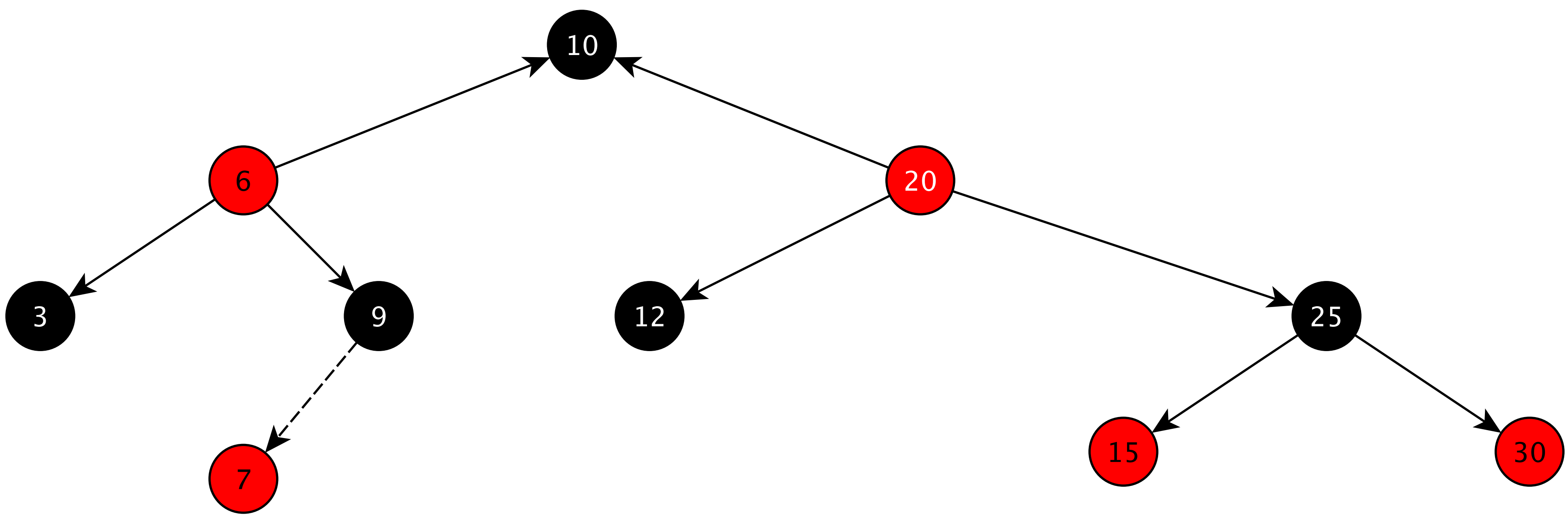
Left rotation at 6, about to do right rotation

Red-Black Tree Insertion



Right rotation at 20, black height broken, need tor recolor

Red-Black Tree Insertion



Color conditions restored, black-height restored.