CSCI 136
Data Structures &
Advanced Programming

Lecture 25

Fall 2017

Instructor: B^2

Last Time

- Binary search trees (Ch 14)
  - The *locate* method
  - Further Implementation

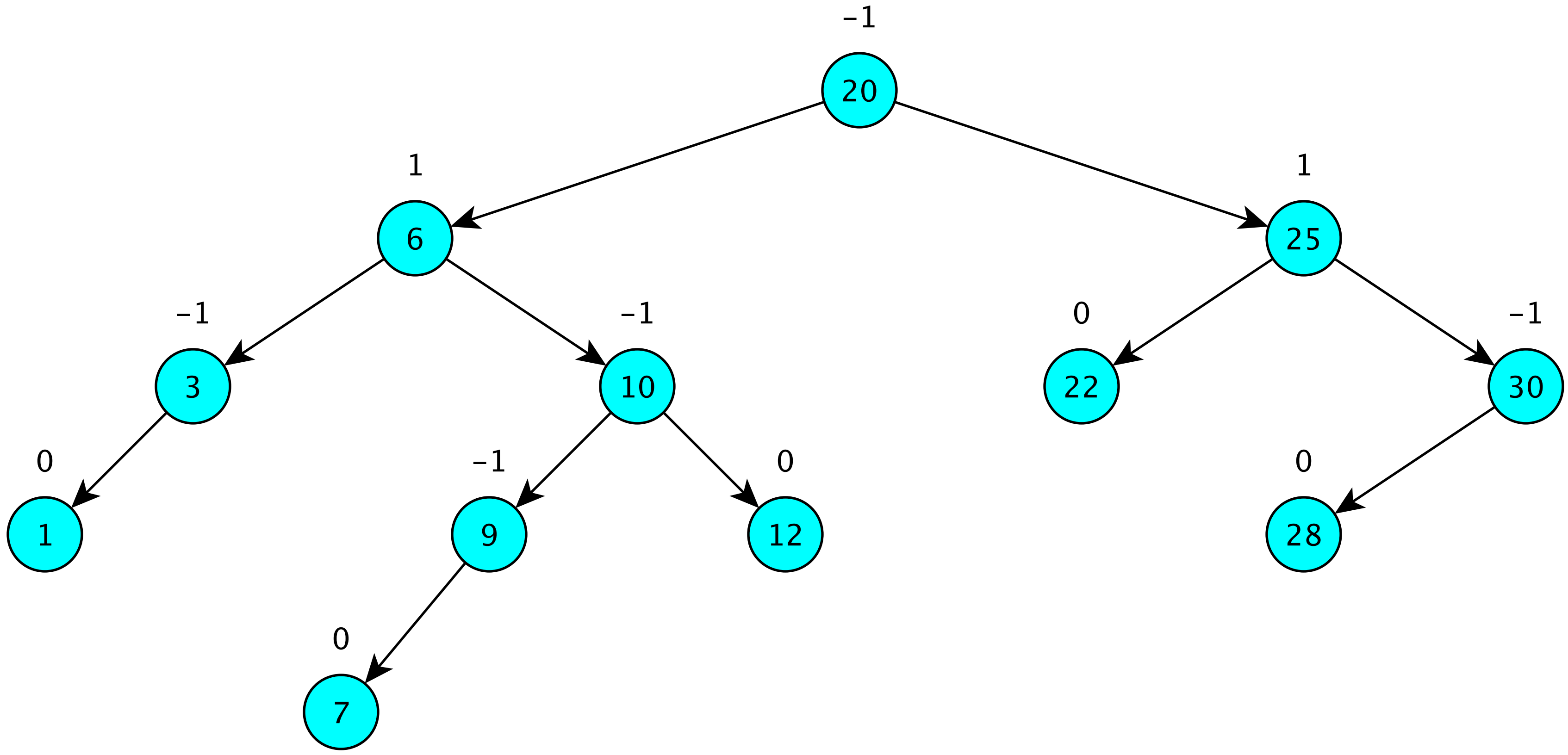
Today's Outline

- Binary search trees (Ch 14)
  - Tree balancing to maintain small height
    - AVL Trees
  - Partial taxonomy of balanced tree species
  - Red-Black Trees
  - Splay Trees

But What About Height?

- Can we design a binary search tree that is always “shallow”?
- Yes! In many ways. Here’s one
- AVL trees
  - Named after its two inventors, G.M. Adelson-Velsky and E.M. Landis, who published a paper about AVL trees in 1962 called "An algorithm for the organization of information"

AVL Trees



AVL Trees

- Balance Factor of a binary tree node:
  - height of right subtree minus height of left subtree.
  - A node with balance factor 1, 0, or -1 is considered *balanced*.
  - A node with any other balance factor is considered unbalanced and requires rebalancing the tree.
- Definition: An *AVL* Tree is a binary tree in which every node is balanced.

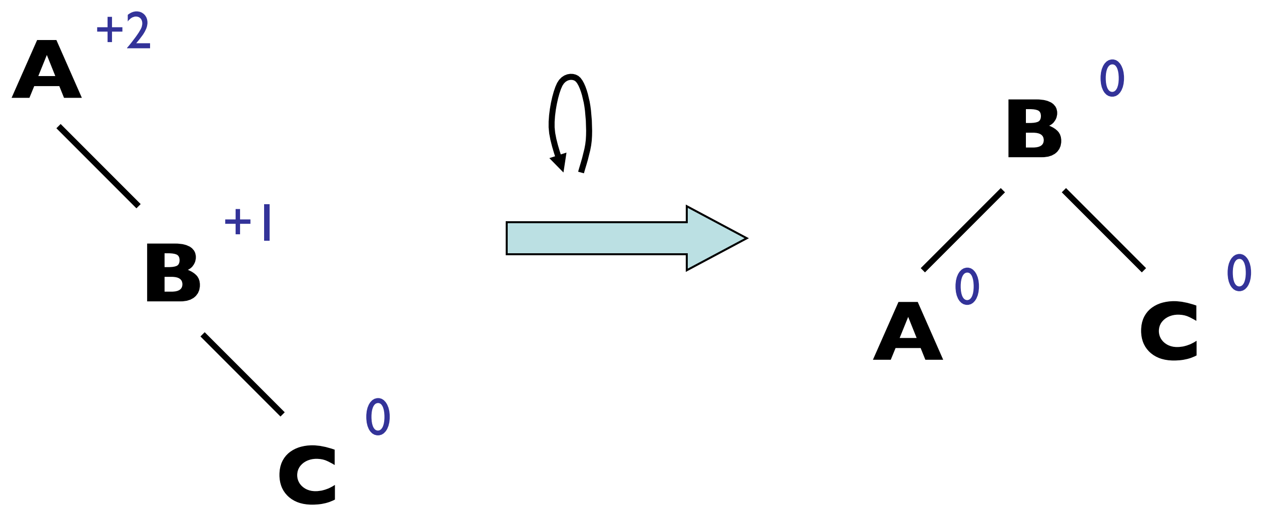
AVL Trees have O(log n) Height

Theorem: An AVL tree on n nodes has height O(log n)

Proof idea

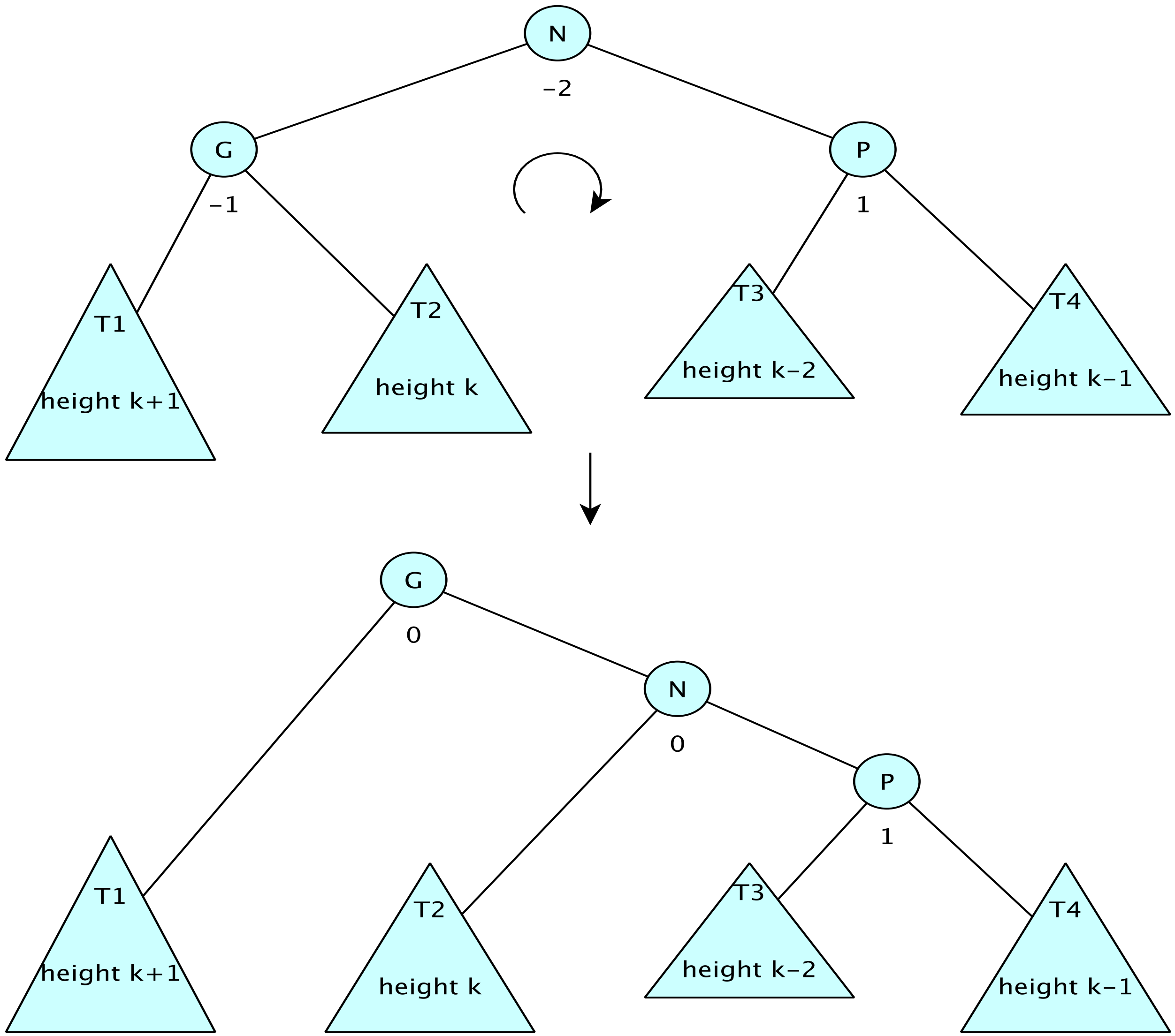
- Show that an AVL tree of height h has at least fib(h) nodes (easy induction proof---try it!)
- Recall (HW): $fib(h) \geq \left({}^{3}/_{2}\right)^{h}$ if h ≥ 10
- So $n \geq \left({}^{3}/_{2}\right)^{h}$ and thus $\log_{3/2} n \geq h$
  - Recall that for any $a, b > 0$, $\log_a n = \frac{\log_b n}{\log_b a}$
  - So $\log_a n$ and $\log_b n$ are Big-O of one another
- So h is O(log n)

Single Rotation

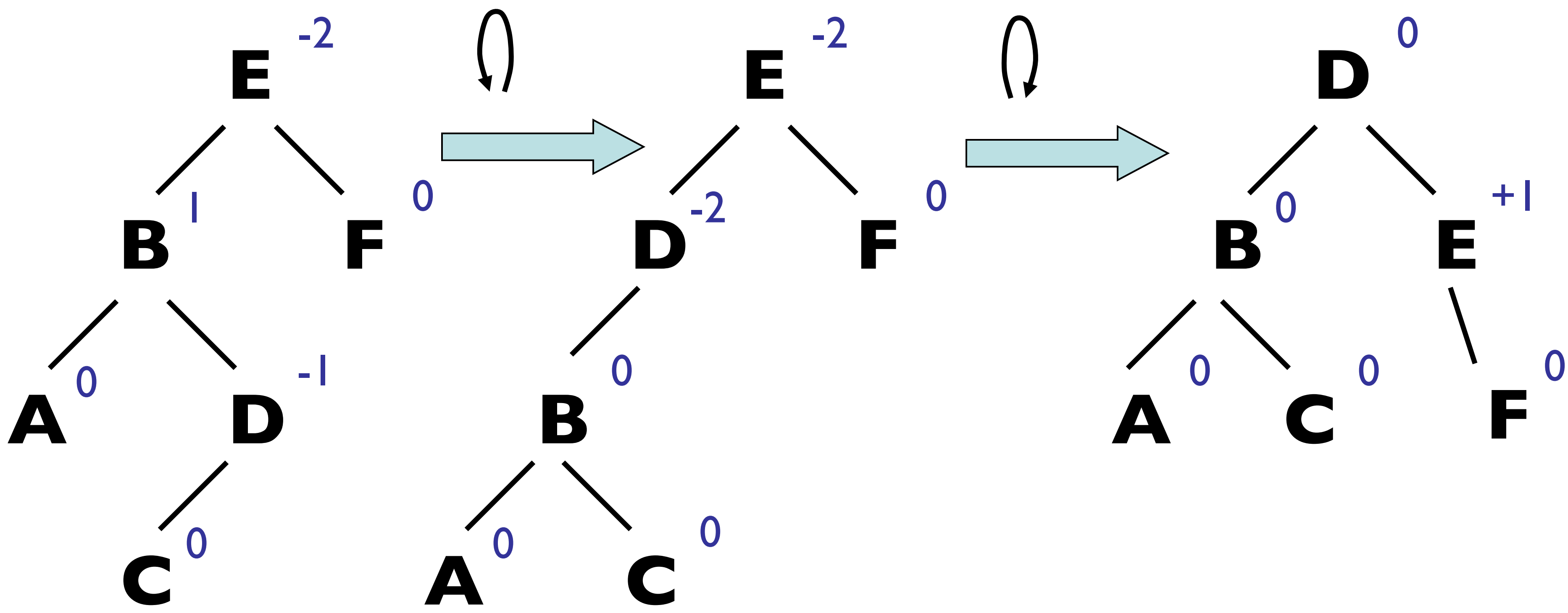


Unbalanced trees can be rotated to achieve balance.

Single Right Rotation



Double Rotation



AVL Tree Facts

- A tree that is AVL except at root, where root balance factor equals ± 2 can be rebalanced with at most 2 rotations
- add(v) requires at most O(log n) balance factor changes and one (single or double) rotation to restore AVL structure
- remove(v) requires at most O(log n) balance factor changes and (single or double) rotations to restore AVL structure
- An AVL tree on n nodes has height O(log n)

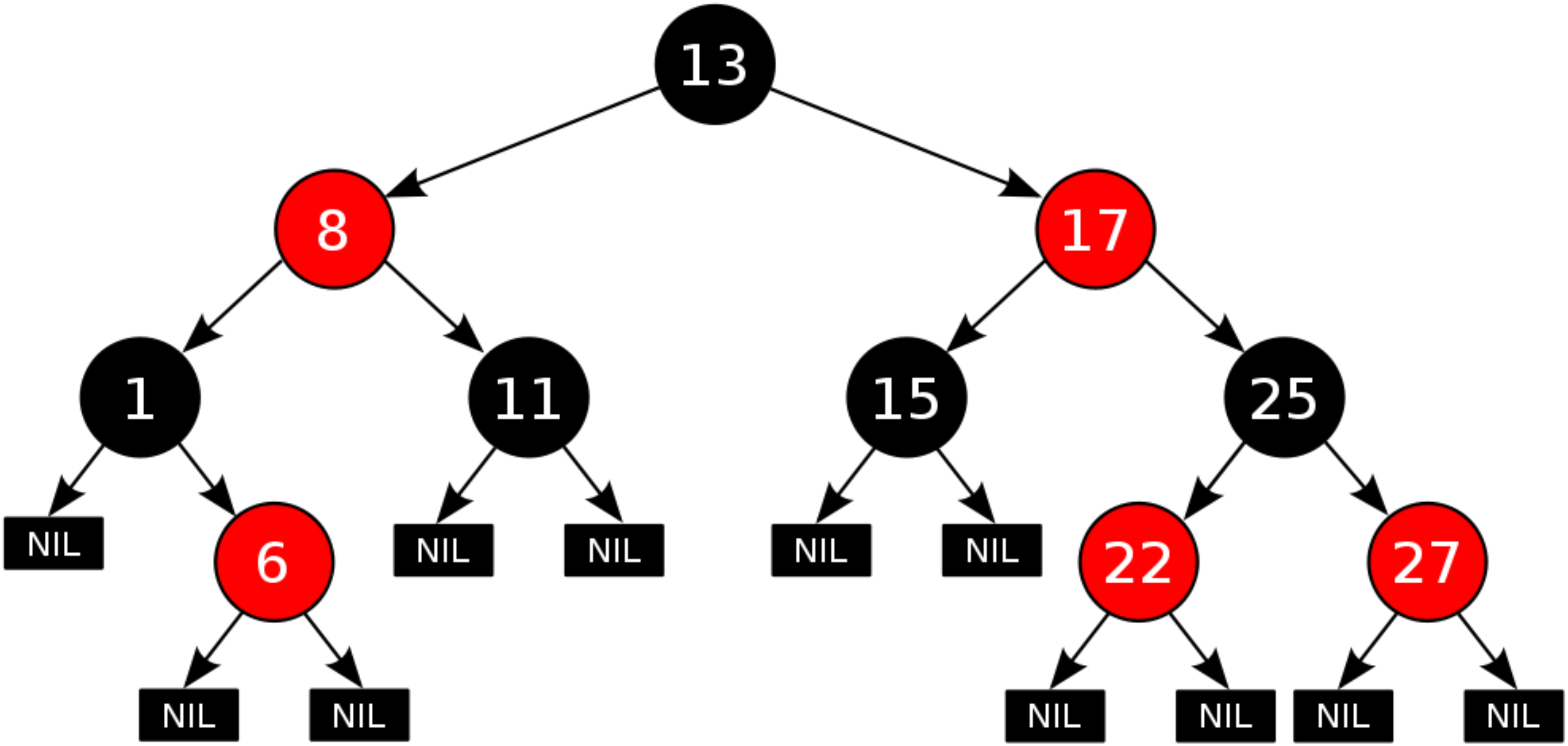
AVL Trees: One of Many

There are many strategies for tree balancing to preserve O(log n) height, including

- AVL Trees: guaranteed O(log n) height
- Red-black trees: guaranteed O(log n) height
- B-trees (not binary): guaranteed O(log n) height
  - 2-3 trees, 2-3-4 trees, red-black 2-3-4 trees, ...
- Splay trees: *Amortized* O(log n) time operations
- Randomized trees: O(log n) expected height

A Red-Black Tree

(from Wikipedia.org)



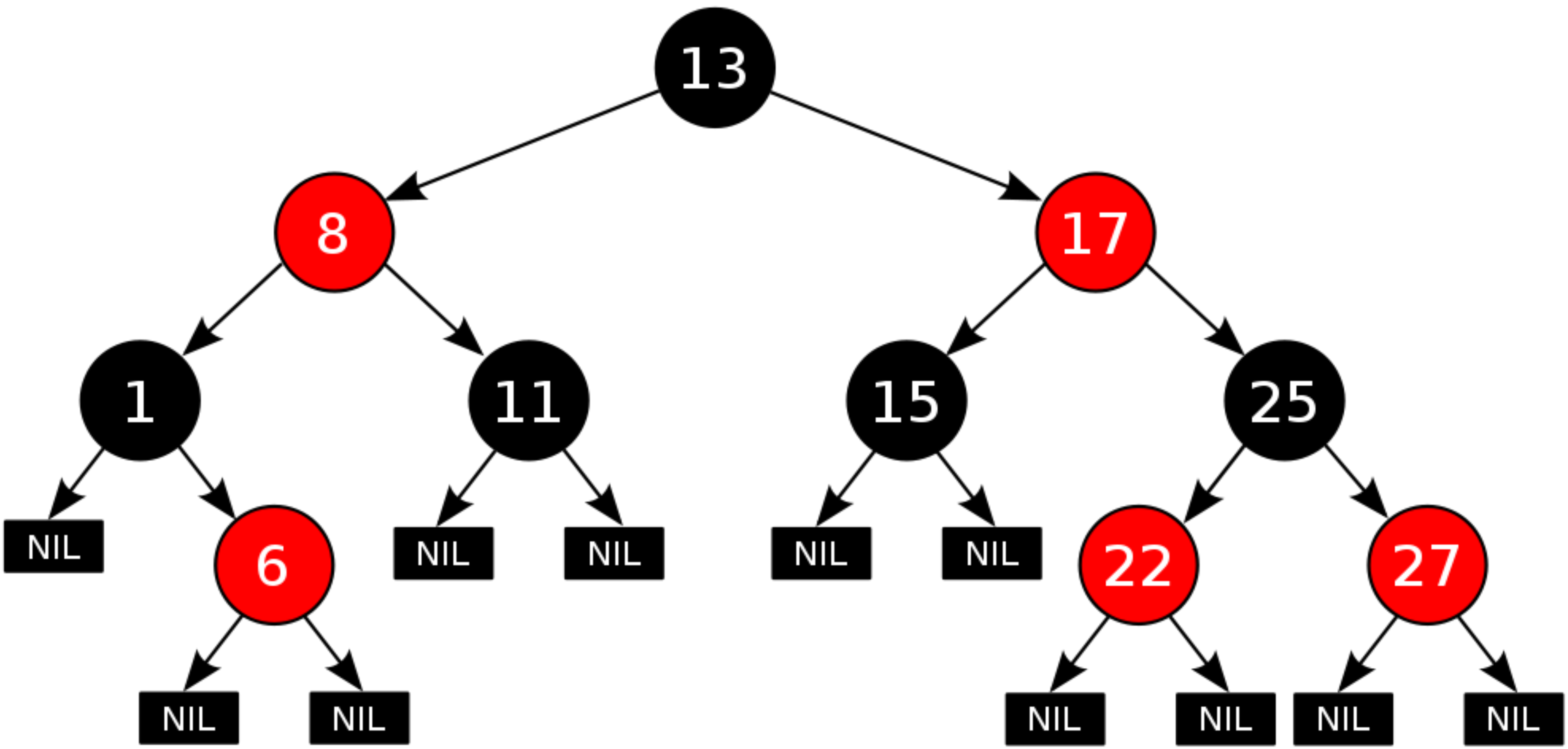
Red-Black Trees

Red-Black trees, like AVL, guarantee shallowness

- Each node is colored *red* or *black*
- Coloring satisfies these rules
  - All empty trees are black
    - We consider them to be the leaves of the tree
  - Children of red nodes are black
  - All paths from a given node to it's descendent leaves have the *same number* of black nodes
    - This is called the *black height* of the node

A Red-Black Tree

(from Wikipedia.org)



Red-Black Trees

The coloring rules lead to the following result

Proposition: No leaf has depth more than twice that of any other leaf.

This in turn can be used to show

Theorem: A Red-Black tree with n internal nodes has height satisfying $h \leq 2\log(n+1)$

- Note: The tree will have *exactly* n+1 (empty) leaves
  - since each internal node has two children

Red-Black Trees

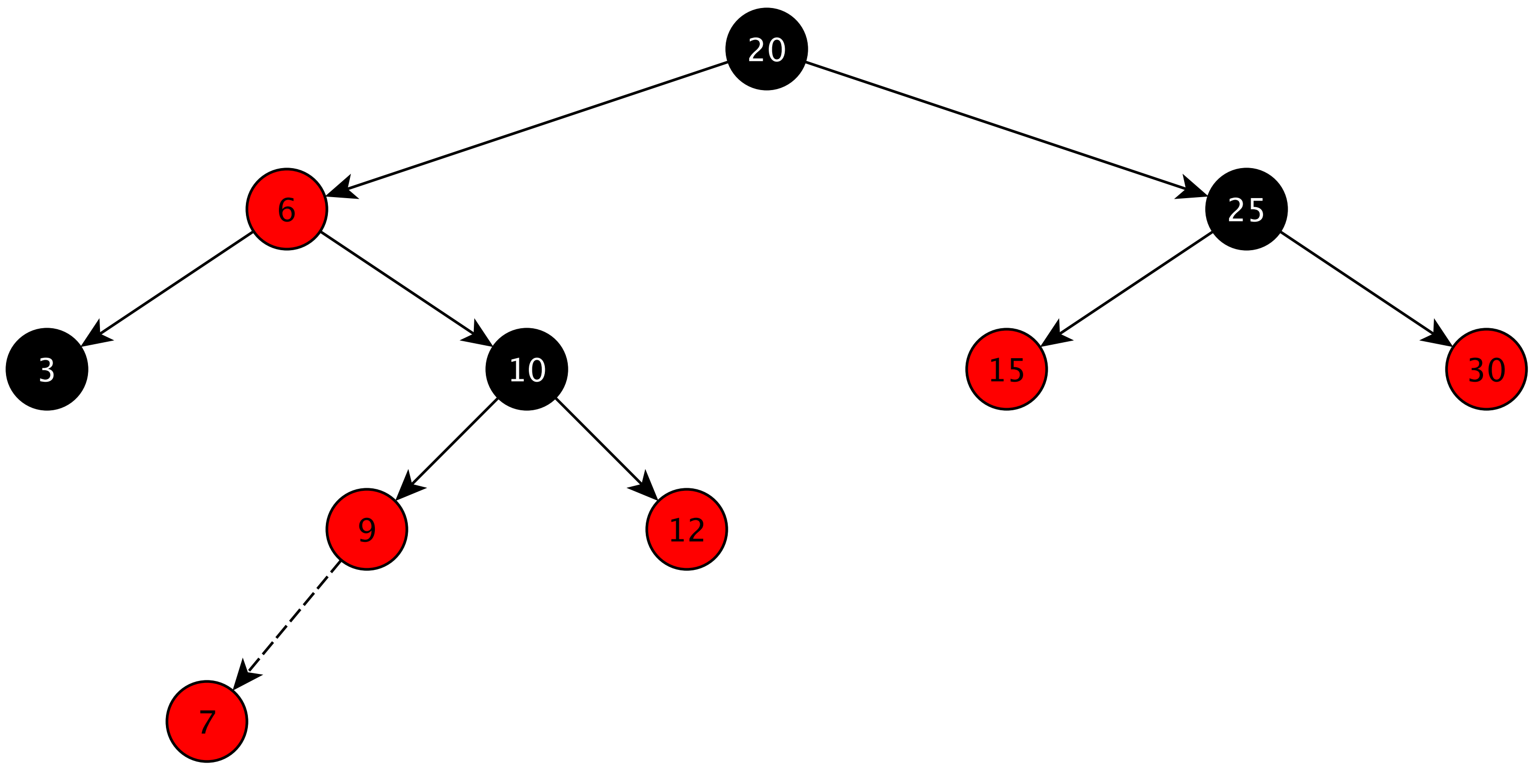
Theorem: A Red-Black tree with n *internal* nodes has height satisfying $h \leq 2\log(n+1)$

Proof sketch: Note: we count empty tree nodes!

- If root is red, recolor it black.
- Now merge red children into (black) parents
  - Now n' ≤ n nodes and height h' ≥ h/2
- New tree has all children with degree 2, 3, or 4
  - All leaves have depth *exactly* h' and there are n+1 leaves
    - So $n+1 \geq 2^{h'}$, so $\log_2(n+1) \geq h' \geq \frac{h}{2}$
- Thus $2\log_2(n+1) \geq h$

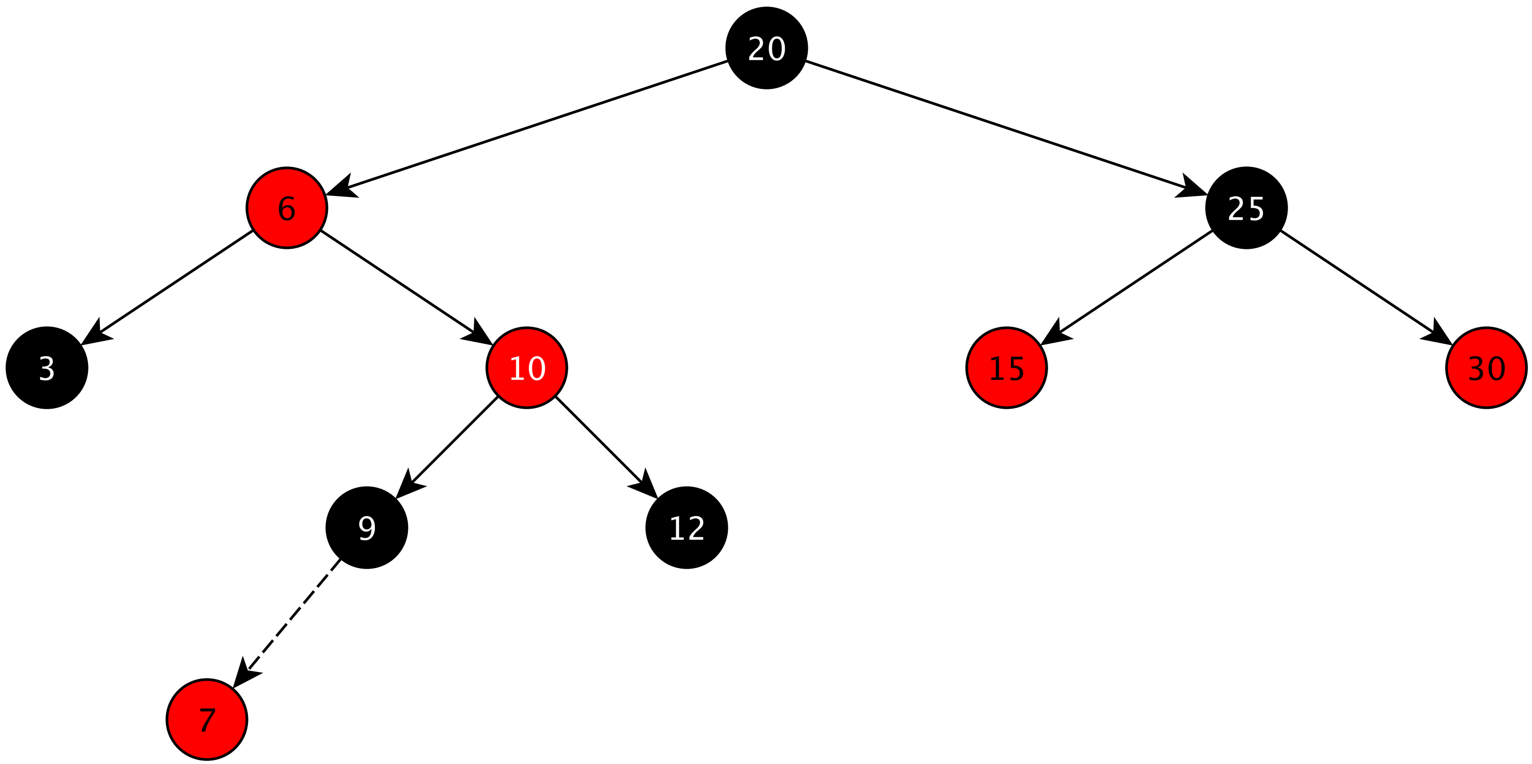
Corollary: R-B trees with n nodes have height O(log n)

Red-Black Tree Insertion



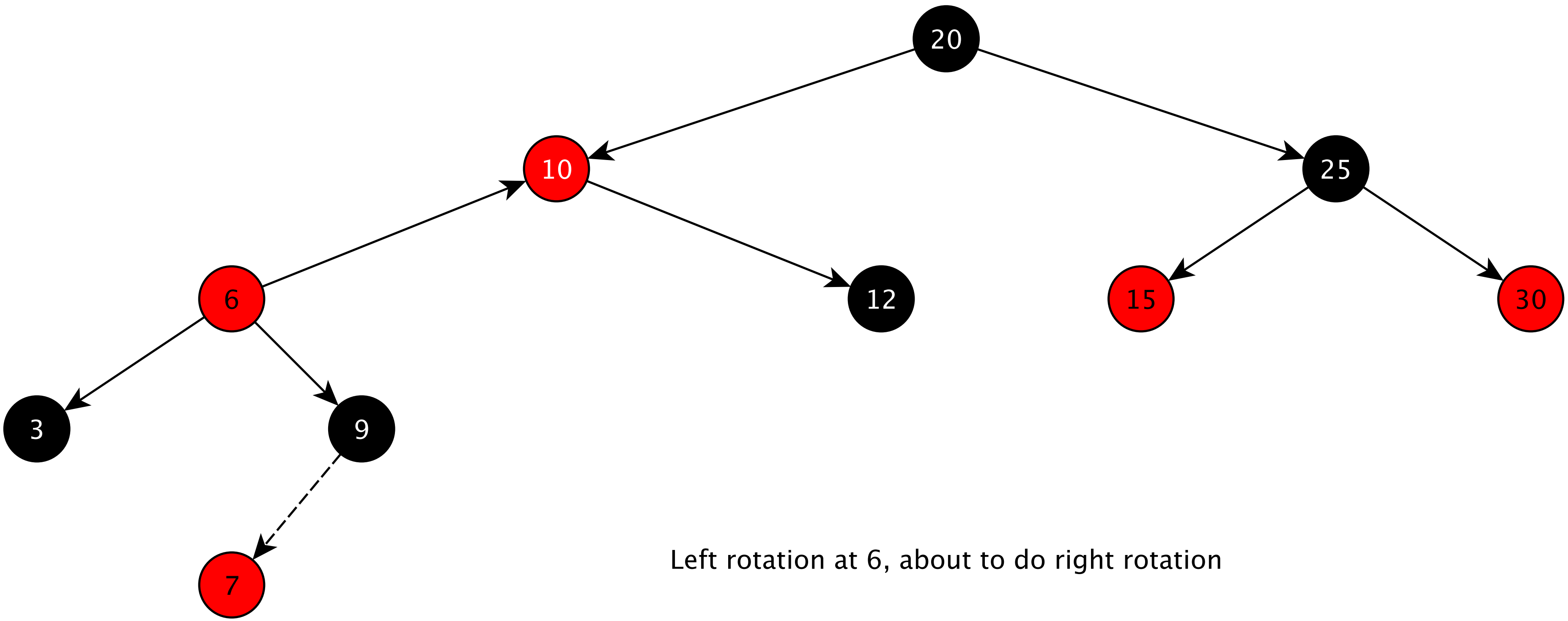
Black empty leaves not drawn. 7 just added Black-height still 2.

Red-Black Tree Insertion



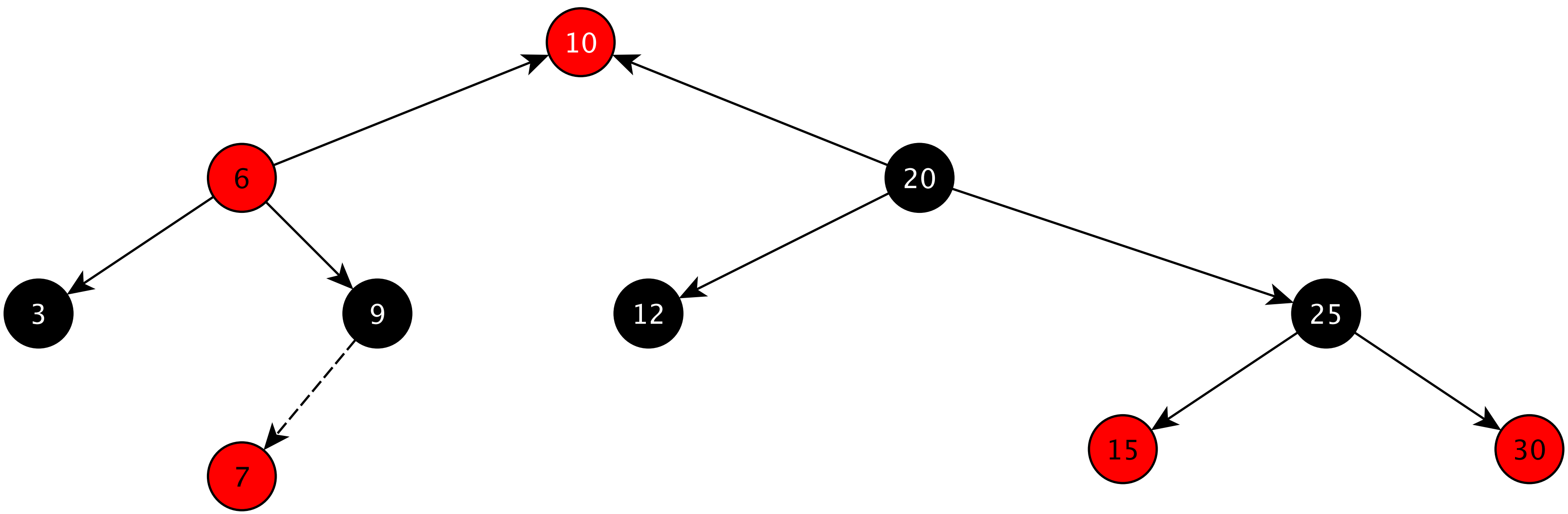
Black height still 2, color violation moved up

Red-Black Tree Insertion



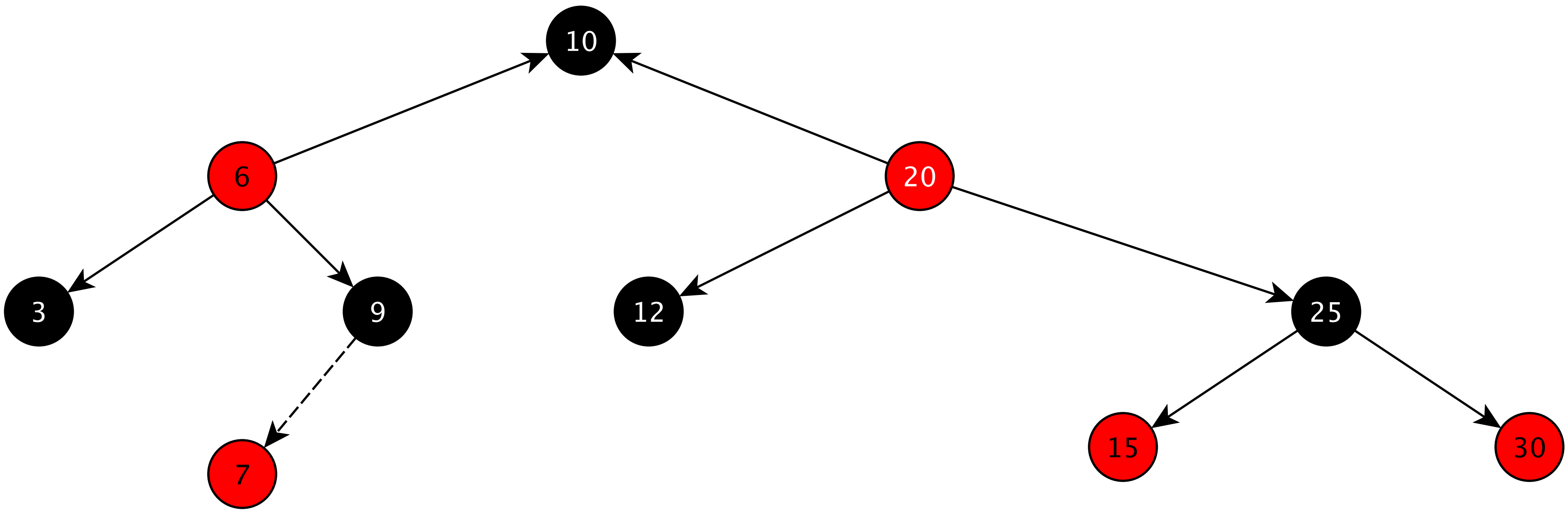
Left rotation at 6, about to do right rotation

Red-Black Tree Insertion



Right rotation at 20, black height broken, need to recolor

Red-Black Tree Insertion



Color conditions restored, black-height restored.

Splay Trees

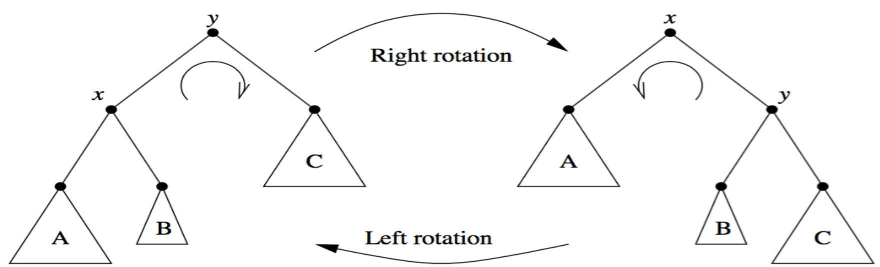
Splay trees are self-adjusting binary trees

- Each time a node is accessed, it is moved to root position via rotations
- No guarantee of balance (or shallow height)
- But good *amortized* performance

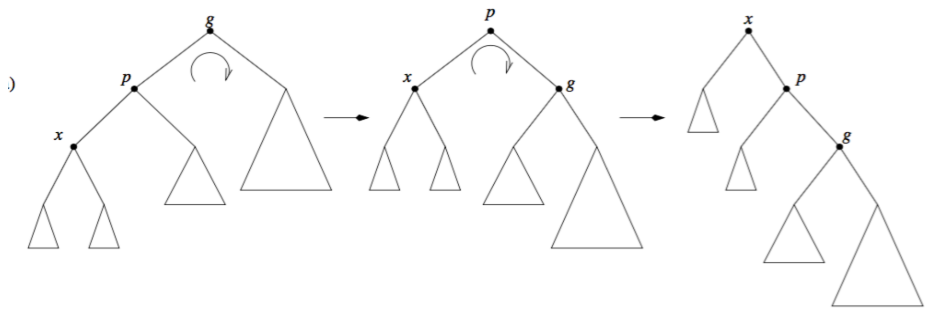
Theorem: Any set of m operations (add, remove, contains, get) on an n-node splay tree take at most O(m log n) time.

Splay Tree Rotations

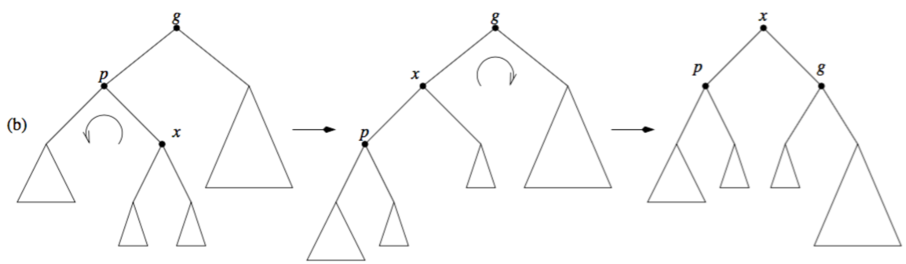
Right Zig Rotation (left version too)



Right Zig-Zig Rotation (left version too)



Right Zig-Zag Rotation (left version too)



Splay Tree Iterator

- Even *contains* method changes splay tree shape
- This breaks the standard in-order iterator!
  - Because the stack is based on the shape of the tree
- Solution: Remove the stack from the iterator
- Observation: Given location of current node (node whose value is next to be returned), we can compute it's (in-order)successor in *next()*
  - It's either left-most leaf of right child of current, or
  - It's closest "left-ancestor" of current
    - Ancestor whose left child is also an ancestor of current
- Also, reset must "re-find" root
  - Idea: Hold a single "reference" node, use it to find root