CSCI 136
Data Structures &
Advanced Programming

Lecture 19

Fall 2017

Instructor: Bills

Last Time:

- Ordered Structures
- Trees
  - Structure, Terminology, Examples

Today

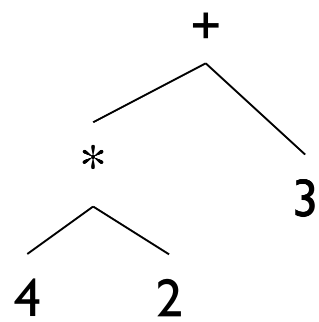
- Trees
  - Implementation
  - Recursion/Induction on Trees
  - Applications
  - Traversals

Introducing Binary Trees

- Degree of each node at most 2
- Recursive nature of tree
  - Empty
  - Root with left and right subtrees
- SLL: Recursive nature was captured by hidden node (Node<E>) class
- Binary Tree: No “inner” node class; single BinaryTree class does it all
- Not part of Structure hierarchy!

Expression Trees

4 * 2 + 3



Build using constructor

```
new BinaryTree<E>(value, leftSubTree, rightSubTree)
```

```
BinaryTree<String> fourTimesTwo =  new BinaryTree<String>
        (“*”,new BinaryTree<String>(“4”),new BinaryTree<String>(“2”));

BinaryTree<String> fourTimesTwoPlusThree = new BinaryTree<String>
        (“+”, fourTimesTwo,  new BinaryTree<String>(“3”));
```

Expression Trees

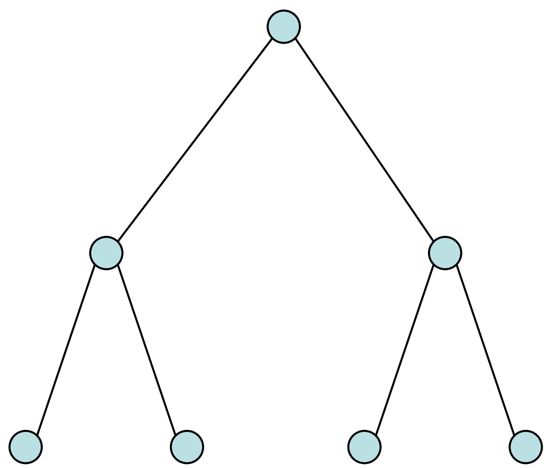
- General strategy
  - Make a binary tree (BT) for each leaf node
  - Move from bottom to top, creating BTs
  - Eventually reach the root
  - Call "evaluate" on final BT

- Example
  - How do we make a binary expression tree for (((4+3)*(10-5))/2)
    - Postfix notation: 4 3 + 10 5 - * 2 /

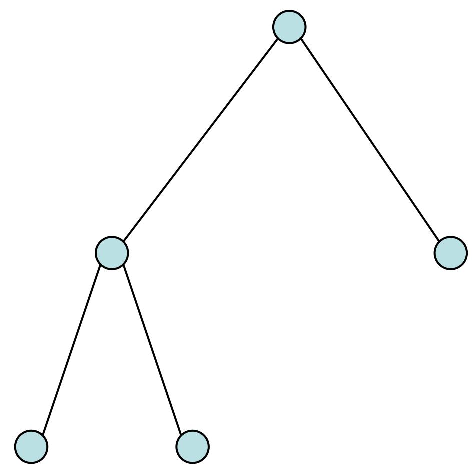
```
int evaluate(BinaryTree<String> expr) {

    if (expr.height() == 0)
      return Integer.parseInt(expr.value());

    else {
      int left = evaluate(expr.left());
      int right = evaluate(expr.right());
      String op = expr.value();
      switch (op) {
      case "+" : return left + right;
      case "-" : return left - right;
      case "*" : return left * right;
      case "/" : return left / right;
      }
      Assert.fail("Bad op");
      return -1;
   }
}
```

Full vs. Complete (non-standard!)

- Full tree – A full binary tree of height h has *leaves only* on level h, and each internal node has exactly 2 children.



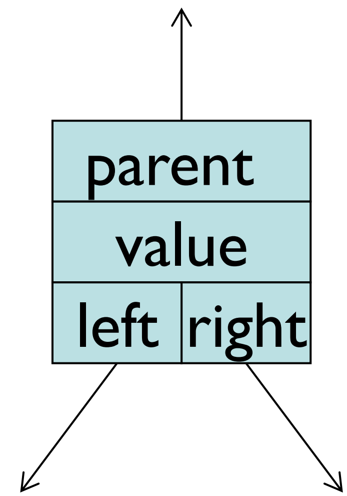
- Complete tree – A *complete* binary tree of height h is *full* to height h-1 and has all leaves at level h in leftmost locations.

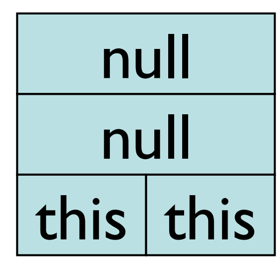


All full trees are complete, but not all complete trees are full!

Implementing BinaryTree

- BinaryTree<E> class
  - Instance variables
    - BinaryTree: parent, left, right
    - E: value
- left and right are never null
  - If no child, they point to an "empty" tree
    - Empty tree T has value null, parent null, left = right = T
  - Only empty tree nodes have null value

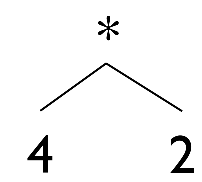


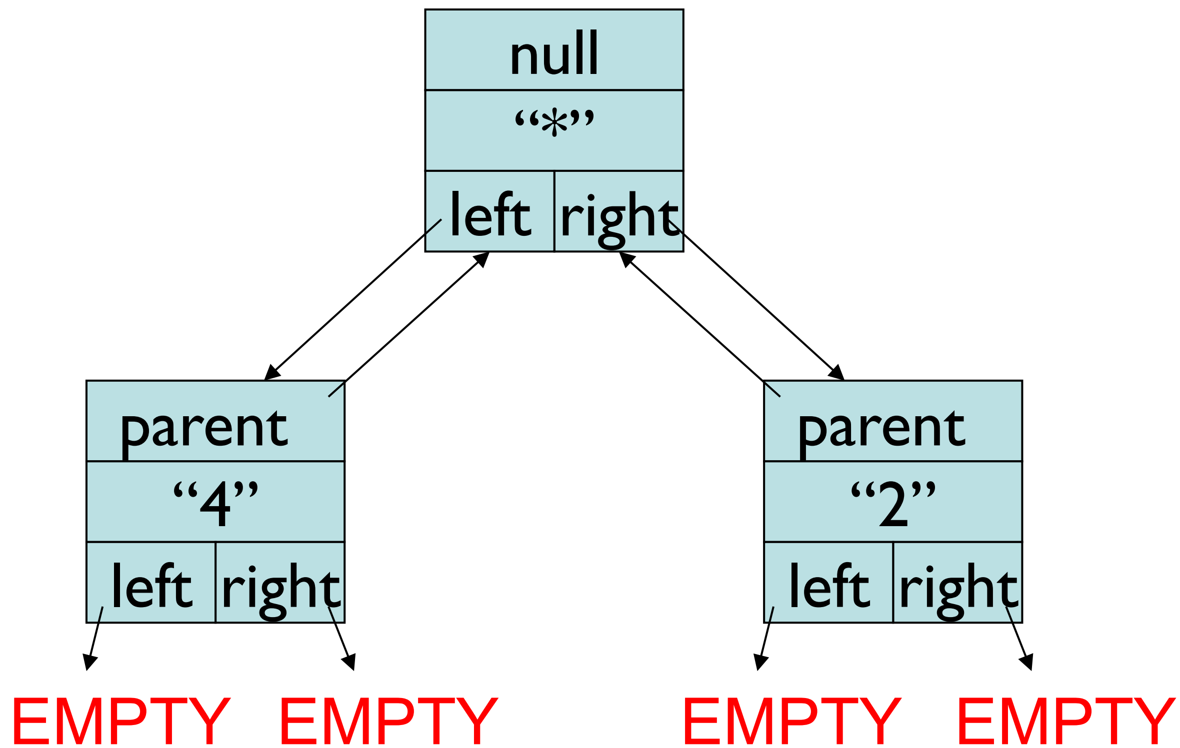


EMPTY BT

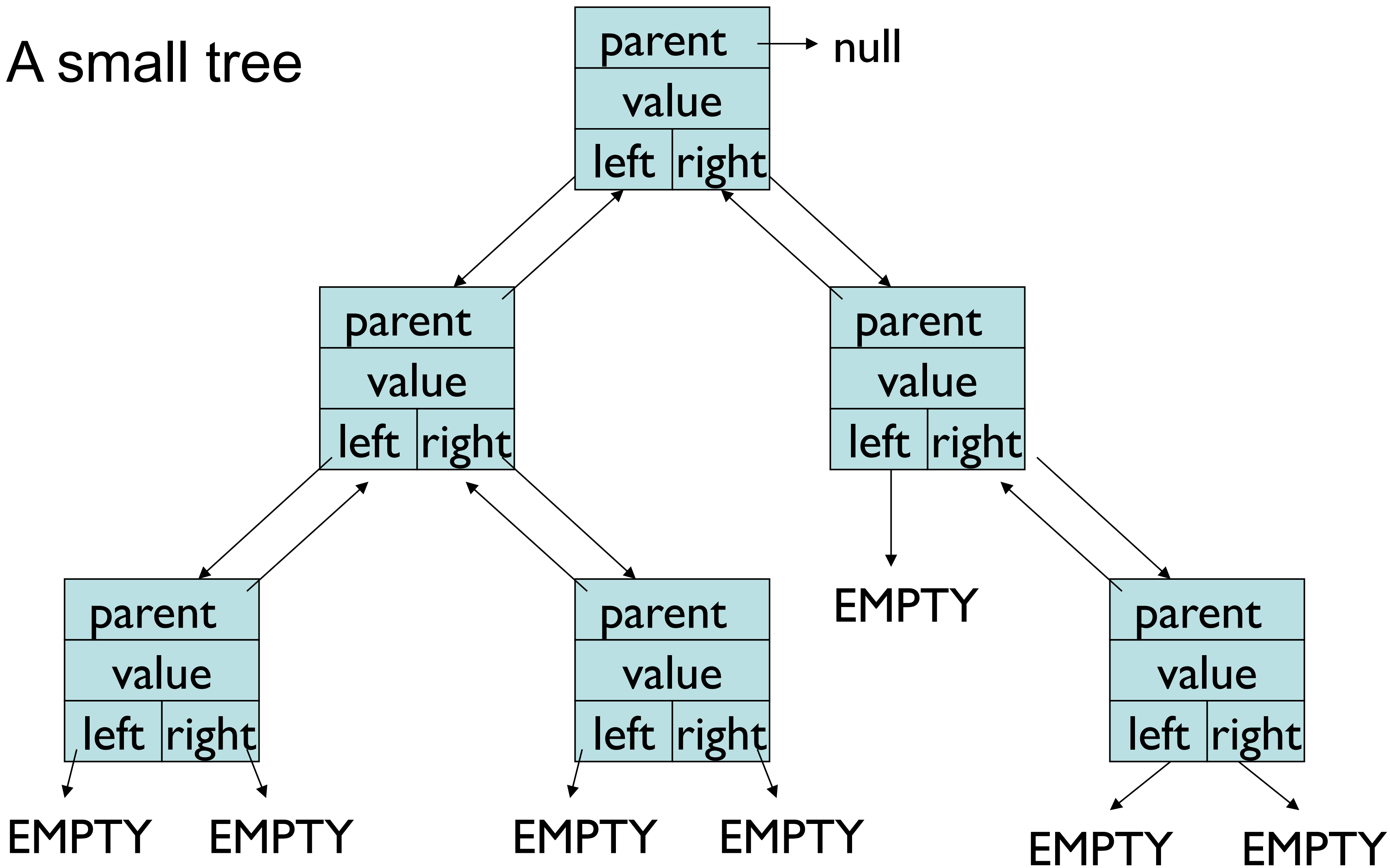
Implementing BinaryTree

- BinaryTree class
  - Instance variables
    - BT parent, BT left, BT right, E value





A small tree



EMPTY != null!

Implementing BinaryTree

- Many (!) methods: See BinaryTree javadoc page
- All "left" methods have equivalent "right" methods
  - public BinaryTree()
    - // generates an empty node (EMPTY)
    - // parent and value are null, left=right=this
  - public BinaryTree(E value)
    - // generates a tree with a non-null value and two empty (EMPTY) subtrees
  - public BinaryTree(E value, BinaryTree<E> left, BinaryTree<E> right)
    - // returns a tree with a non-null value and two subtrees
  - public void setLeft(BinaryTree<E> newLeft)
    - // sets left subtree to newLeft
    - // re-parents newLeft by calling newLeft.setParent(this)
  - protected void setParent(BinaryTree<E> newParent)
    - // sets parent subtree to newParent
    - // called from setLeft and setRight to keep all "links" consistent

Implementing BinaryTree

- Methods:
  - public BinaryTree<E> left()
    - // returns left subtree
  - public BinaryTree<E> parent()
    - // post: returns reference to parent node, or null
  - public boolean isLeftChild()
    - // returns true if this is a left child of parent
  - public E value()
    - // returns value associated with this node
  - public void setValue(E value)
    - // sets the value associated with this node
  - public int size()
    - // returns number of (non-empty) nodes in tree
  - public int height()
    - // returns height of tree rooted at this node
  - But where's "remove" or "add"?!?!

BT Questions/Proofs

- Prove
  - The number of nodes at depth n is at most 2^n.
  - The number of nodes in tree of height n is at most $2^{(n+1)}-1$.
  - A tree with n nodes has exactly n-1 edges
  - The size() method works correctly

BT Questions/Proofs

Prove: Number of nodes at depth $d \geq 0$ is at most 2^d.

Idea: Induction on depth d of nodes of tree

Base case: $d = 0$: 1 node. $1 = 2^0$ ✓

Induction Hyp.: For some $d \geq 0$, there are at most 2^d nodes at depth d.

Induction Step: Consider depth $d+1$. It has at most 2 nodes for every node at depth d.

Therefore it has at most $2 * 2^d = 2^{d+1}$ nodes ✓

BT Questions/Proofs

Prove that any tree of $n \geq 1$ nodes has $n-1$ edges

Idea: Induction on number of nodes

Base case: $n = 1$. There are no edges ✓

Induction Hyp: Assume that, for some $n \geq 1$, every tree of n nodes has exactly $n-1$ edges.

Induction Step: Let T have $n+1$ nodes. Show it has exactly n edges.

- Remove a leaf v (and its single edge) from T
- Now T has n nodes, so it has $n-1$ edges
- Now add v (and its single edge) back, giving $n+1$ nodes and n edges.

BT Questions/Proofs

Alternate Proof: Strong Induction

Induction Hyp.: For some $n \geq 1$, every tree T with $k \leq n$ nodes has exactly $k-1$ edges.

Induction Step: Let T have $n+1$ nodes

- Let $n(T)$ = # of nodes of T and $e(T)$ = # of edges of T
- Remove the root node r of T along with its 2 edges
- This leaves the two subtrees T_L and T_R of T
- T_L and T_R each have at most n nodes
- So $n(T_L) = 1 + e(T_L)$ and So $n(T_R) = 1 + e(T_R)$
- Now add r (and its 2 edges) back
  - Then $n(T) = 1 + n(T_L) + n(T_R)$ and $e(T) = 2 + e(T_L) + e(T_R)$
  - But $n(T_L) + n(T_R) = 1 + e(T_L) + 1 + e(T_R) = e(T)$ ✓

Special case: One of T_L or T_R is empty. What changes?

BT Questions/Proofs

Prove that BinaryTree method size() is correct.

- Let n be the number of nodes in the tree T
- Alert: Strong Induction Ahead...

Base case: n = 0. T is empty---size() returns 0✓

Induction Hyp: Assume size() is correct for *all trees* having *at most* n nodes.

Induction Step: Assume T has n+1 nodes

- Then left/right subtrees each have *at most* n nodes
- So size() returns correct value for each subtree
- And the size of T is 1 + size of left subtree + size of right subtree✓