Lecture 8

Homework #8: 2.2.6, 2.2.7, 2.2.9, 2.2.10. Give an NFA that accepts the language $(a \cup b)^*ab^+a(a \cup b)ab^*$, and then find and equivalent DFA.

Last time: introduced nondeterministic finite automata.

Now, let's show that **NFAs = DFAs.** [Began this last time, seeing how transitions on strings of length >1 could be eliminated by “stretching” the automaton.]

We'll say that two finite automata M_1 and M_2 are equivalent iff

$L(M_1) = L(M_2)$.

Thm. For each NFA, there is an equivalent DFA.

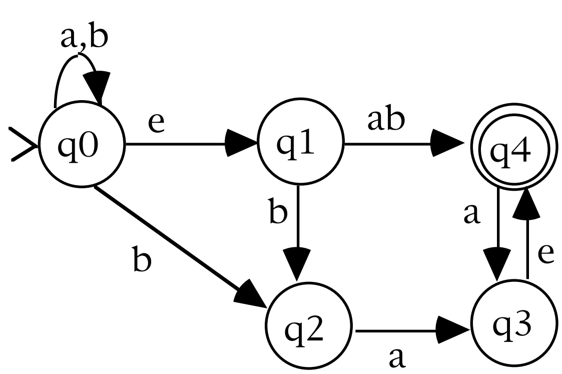
Proof will proceed by construction. That is, we will show how to turn an NFA into a DFA.

To do this, we need to:

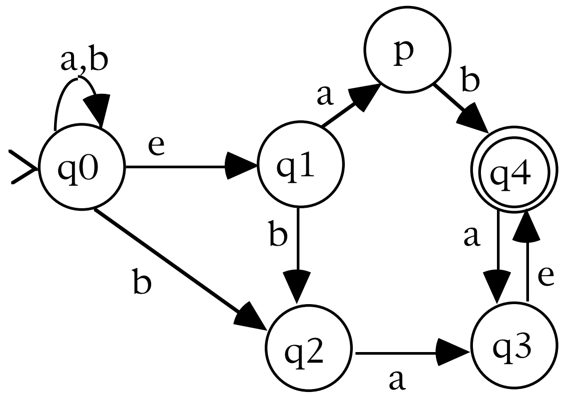
(1) eliminate transitions on e.
(2) eliminate transitions on strings of length > 1.
(3) add transitions to have actions for all symbols in a given state.
(4) eliminate multiple transitions from 1 state.

(2) is easy - just "stretch" the NFA by adding states.

Example.



becomes



(of course, we'd need to prove that the "stretched" NFA is equiv to the original, but it's fairly obvious)

Basically, if $(q,u,q^1) \in \Delta$, and $|u| > 1$, then we can write $u = u_1 u_2 \ldots u_n$.
To "stretch" the NFA, we add transitions to Δ:

$(q,u_1,p_1), (p_1,u_2,p_2), \ldots, (p_{n-1},u_n,q^1)$.

Let's call the initial NFA $M = (K, \Sigma, \Delta, s, F)$, and
the stretched NFA $M^1 = (K^1, \Sigma, \Delta^1, s^1, F^1)$.

Now, for the remainder of the proof:
We will view the NFA as follows: **at any point in time, it can be in many states at once.**

Example cont'd. On input ab, it can be in any of {q0, q1, q2, q4}
We can view this as a single state in a DFA.
The idea is that we're building "multi-states".

Now, what are subsequent states?
Anything that can be reached from one of these on a given input symbol.

So, if the next symbol were "a": {q0, q1, p, q3}

Basically, **the DFA is simulating all moves of the NFA simultaneously**.

Now let's look at transitions on e - these need to be considered specially.

Define the states that are **reachable** from a state q **on no input:**

$\mathbf{E(q)} = \{p \in K^1: (q,e) \vdash^* (p,e)\}$

or

$E(q) = \{p \in K^1: (q,w) \vdash^* (p,w)\}$

In our Example.

E(q0) = {q0, q1}
E(q1) = {q1}
E(q2) = {q2}
E(q3) = {q3, q4}
E(q4) = {q4}
E(p) = {p}

Before giving the formal construction, let's take these ideas and complete the construction of a DFA from the NFA above:

S" = {q0, q1}

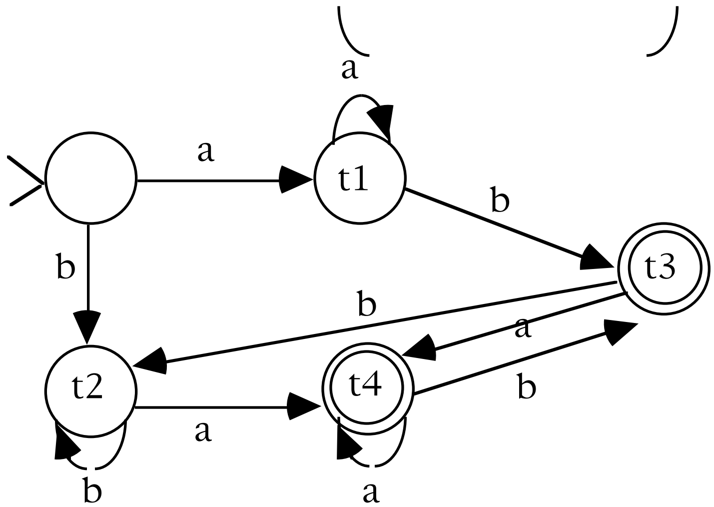
δ"(S", a) = E(q0) $\cup$ E(p) = {q0, q1, p} = t1
δ"(S", b) = E(q0) $\cup$ E(q2) = {q0, q1, q2} = t2

δ"(t1, a) = E(q0) $\cup$ E(p) = {q0, q1, p} = t1
δ"(t1, b) = E(q0) $\cup$ E(q2) $\cup$ E(q4)= {q0, q1, q2, q4} = t3 (FINAL STATE)

δ"(t2, a) = E(q0) $\cup$ E(p) $\cup$ E(q3) = {q0, q1, p, q3, q4} = t4 (FINAL)
δ"(t2, b) = E(q0) $\cup$ E(q2) = {q0, q1, q2} = t2

δ"(t3, a) = E(q0) $\cup$ E(p) $\cup$ E(q3) = {q0, q1, p, q3, q4} = t4
δ"(t3, b) = E(q0) $\cup$ E(q2) = {q0, q1, q2} = t2

δ"(t4, a) = E(q0) $\cup$ E(p) $\cup$ E(q3) = {q0, q1, p, q3, q4} = t4
δ"(t4, b) = E(q0) $\cup$ E(q2) $\cup$ E(q4) = {q0, q1, q2, q4} = t3



Now, let's state the construction of the DFA more formally:

We construct a DFA $M'' = (K'', \Sigma, \delta'', s'', F'')$, where

$K'' = 2^{K'}$ (power set - i.e., the set of all sets of states in M^1)

$s'' = E(s^1)$

$F'' = \{Q \subseteq K^1 : Q \cap F^1 \neq \varnothing\}$

$\delta''(Q,\sigma) = \cup\{E(p) : p \in K^1, (q,\sigma,p) \in \Delta^1, q \in Q\}$
(a transition records all moves on a symbol, including adjacent e-transitions)

In our Example. $s'' = \{q0, q1\}$

Now, we need to prove that

I. This is a **deterministic** FA
II. It is **equivalent to M^1**

I. This part is easy - by our definition of δ, M'' is deterministic.

II. Claim: for any $w \in \Sigma^*$, and $q, p \in K^1$

$(q,w) \vdash^*_{M^1} (p,e)$ iff $(E(q),w) \vdash^*_{M''} (P,e)$, $p \in P$.

We will prove this by induction on the length of w:

Basis. $|w| = 0$, so $w = e$.

so we need to show
$(q,e) \vdash^*_{M^1} (p,e)$ iff $(E(q),e) \vdash^*_{M''} (P,e)$, $\quad p \in P$.

($\Leftarrow$) Since M" is deterministic, $E(q) = P$, and $p \in P$,
so $(q,e) \vdash^*_{M^1} (p,e)$, by the definition of $E(q)$.

($\Rightarrow$) if $(q,e) \vdash^*_{M^1} (p,e)$, then $p \in E(q)$, according to the definition of E, but then $(E(q),e) \vdash^*_{M''} (E(q),e) = (P,e)$,
$p \in P$.

Induction Step. Assume that the claim is true for all w, $|w| \leq k$, $k \geq 0$.

Show that it also holds for w, $|w| = k + 1$.

Let $w = va$, $v \in \Sigma^*$, $a \in \Sigma$.

($\Rightarrow$) Suppose that $(q,w) \vdash^*_{M^1} (p,e)$, which we can rewrite as
$(q,va) \vdash^*_{M^1} (p,e)$

$= (q, va) \vdash^*_{M^1} (r^1, a) \vdash_{M^1} (r^2, e) \vdash^*_{M^1} (p,e)$

Now, what can we say about the pieces of this computation?

First, rather than considering $(q, va) \vdash^*_{M^1} (r^1, a)$, let's think about $(q, v) \vdash^*_{M^1} (r^1, e)$. The induction hypothesis tells us that

$(E(q),v) \vdash^*_{M''} (R^1,e)$, $r^1 \in R^1$.

Second, since $(r^1, a) \vdash_{M^1} (r^2, e)$
(r^1, a, r^2) is in Δ^1
so $E(r^2) \subseteq \delta''(R^1, a)$, by definition.

Third, since $(r^2, e) \vdash^*_{M^1} (p,e)$, $p \in E(r^2)$.

Putting these all together:

$p \in E(r^2) \subseteq \delta''(R^1, a)$, so $(R^1,a) \vdash (P,e)$, $\quad p \in P$.

So

$(E(q), va) \mid\!\!-\!\!\!-^*_{M''} (R^1, a) \mid\!\!-\!\!\!- (P, e), \qquad p \in P.$

so

$(E(q), va) \mid\!\!-\!\!\!-^*_{M''} (P, e), \quad p \in P.$

The other direction is easier - we won't do it in class.