Lecture 15

Homework #16: 3.1.10 (a), (c), (d)
[For (c) and (d), give constructions and sketch proofs of correctness.]

We've discussed the notion that the CFLs have properties that cannot be expressed by regular expressions or handled by finite automata.
(i.e., that there are languages that are not regular but that are context-free) But what exactly is the relationship between the regular and context-free languages?

$$\textbf{Reg lang} \subset \textbf{CFL}$$

We'll show this next.

But first, a definition:

A CFG $G = (V, \Sigma, R, S)$ is **regular** iff $R \subseteq (V\text{-}\Sigma) \times \Sigma^*((V\text{-}\Sigma) \cup \{e\})$

So, if A and B are non-terminals;
and $w \in \Sigma^*$ then rules have the form:

$A \to wB$
or $A \to w$

Example.
Consider a grammar with the rules

$S \to aB$
$B \to aB$
$B \to bB$
$B \to b$

An example of a derivation of a string is:

$S \Rightarrow aB \Rightarrow aaB \Rightarrow aabB \Rightarrow aabbB \Rightarrow aabbaB \Rightarrow aabbab$

$a\,(a \cup b)^*\,b$

Thm. A language is regular iff it is generated by a regular grammar.

To prove this, we'll show that from a regular grammar we can construct a finite automaton and vice versa.

We'll think of the terminal symbols as transition symbols in a FA;
We'll think of non-terminals as states.

($\Rightarrow$) Suppose L is regular. Then L is accepted by a DFA

$M = (K, \Sigma, \delta, s, F)$

Let $G = (V, \Sigma, R, S)$ where

$V = \Sigma \cup K$
$S = s$
$R = \{q \to ap: \delta(q,a) = p\} \cup \{q \to e: q \in F\}$

i.e., each transition is mirrored by a rule.

in M: $(q_0,\sigma_1\sigma_2...\sigma_n) \vdash (q_1,\sigma_2...\sigma_n) \vdash . . . (q_{n-1},\sigma_n) \vdash (q_n,e)$

in G: $q_0 \Rightarrow \sigma_1 q_1 \Rightarrow \sigma_1\sigma_2 q_2 \Rightarrow . . . \sigma_1\sigma_2...\sigma_n q_n \Rightarrow \sigma_1\sigma_2...\sigma_n$

Now we need to show that L(M) = L(G).
We'll show that

$(s,w) \vdash^* (p,e), p \in F$
iff
$s \Rightarrow^* wp$
(and then since $p \to e$
$s \Rightarrow^* wp \Rightarrow w$)

In fact, we'll show the more general result:

$(q_0,w) \vdash^* (p,e), p \in F$
iff
$q_0 \Rightarrow^* wp$
(and then since $p \to e$
$q_0 \Rightarrow^* wp \Rightarrow w$)

Suppose $w \in L(M)$
then $(q_0,w) \vdash^* (p,e), p \in F$.

but then $q_0 \Rightarrow^* wp$

and since $p \rightarrow e$

$$q_0 \Rightarrow^* wp \Rightarrow w$$

So $w \in L(G)$

[Proof by induction done in class]

($\Leftarrow$) Suppose that L is generated by a regular grammar G = (V, Σ, R, S)

Construct a NFA to accept L(G) as follows:
Let M = (K, Σ, Δ, s, F) where

$K = V\text{-}\Sigma \cup \{f\}$
$s = S$
$F = \{f\}$

Note that f is a single accept state, not related to the grammar in any way.]

$$\Delta = \{(A,w,B): A \rightarrow wB \in R;\ A, B \text{ non-term};\ w \in \Sigma^*\}$$
$$\cup \{(A,w,f): A \rightarrow w \in R;\ A \text{ non-term};\ w \in \Sigma^*\}$$

Rather than proving this, let's convert our earlier language into an NFA:

$S \rightarrow aB$
$B \rightarrow aB$
$B \rightarrow bB$
$B \rightarrow b$

$\Delta = \{(S,a,B), (B,a,B), (B,b,B), (B,b,f)\}$