Lecture 10

Homework #10: 2.3.4, 2.3.7 b [from left to right, label the states 2, 1, 3]
(nothing to hand in)

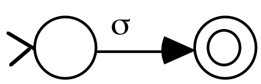
Thm. A language is regular iff it is accepted by a FA.

(⇒) If a language is regular, it is accepted by a finite automaton.

The proof will follow from the definition of a regular language: the regular languages are the smallest class of languages containing ∅ and the singletons {σ}, where σ is a symbol (or e), and closed under union, concatenation, and Kleene Star.

∅ is clearly accepted by a FA:

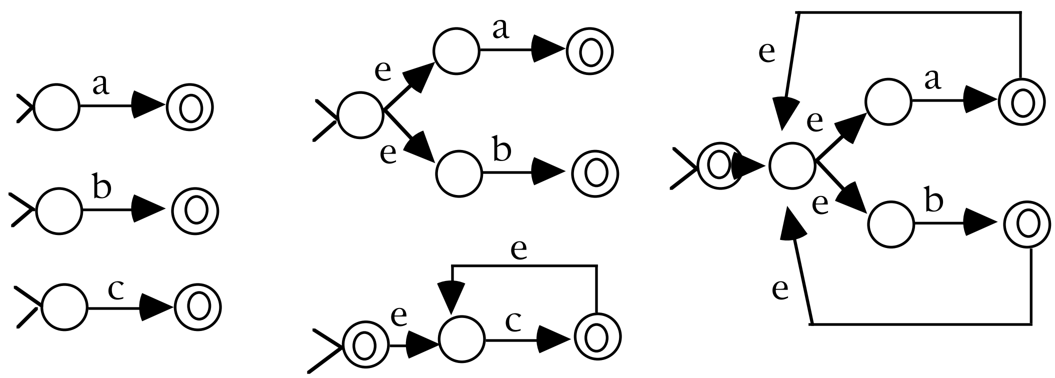
as is {σ}, for any symbol, including e, representing the empty string:

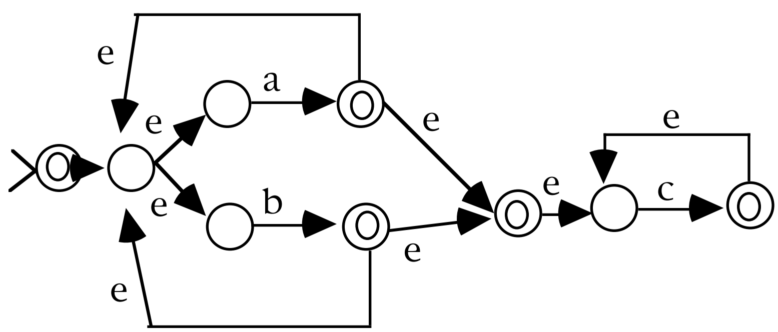


Furthermore, by the theorem last time, the languages accepted by FA are closed under union, concatenation, and Kleene Star.

Thus every regular language is accepted by a FA.

Example. ((a ∪ b)* c*)





($\Leftarrow$) If a language is accepted by a FA, then it is regular.

Assume that M is a DFA (wlog). Show that R = L(M) is regular.

Let M = (K, Σ, δ, s, F). We'll be thinking of L(M) as the union of a number of smaller languages.

Let K = {q1, q2, . . . , qn}, s = q1. [We're just re-naming the states and putting them in an order, with no effect on the actual FA.]

For $1 \leq i,j \leq n$ and $0 \leq k \leq (n)$

Let R(i,j,k) = all strings that drive M from qi to qj without going through states numbered higher than qk. [You can go to it, but not through it.]

That is:

R(i,j,k) = {x ∈ Σ*: (qi,x) |—* (qj,e) and

if (qi,x) |—* (qm,y) then y=e and m=j or
y=x and m=i or
m ≤ k

Now let's specifically consider R(1,j,n) - what does this mean?
It's all the strings that take you from the start state to qj.

Now, what about R(1,j,n) if qj is a final state? It's all strings accepted by reaching qj.

So L(M) = ∪{R(1,j,n): qj ∈ F}

What we'll show is that each of these R(1,j,n) is regular - so their union is as well.

The proof will be (as usual) by induction.

Basis.

$$\{R(i,j,0): qj \in F\} = \{\sigma \in \Sigma: \delta(qi,\sigma) = qj\}, \text{ if } i \neq j$$
$$\text{or } \{e\} \cup \{\sigma \in \Sigma: \delta(qi,\sigma) = qj\}, \text{ if } i = j$$

These are **finite**, and therefore regular.

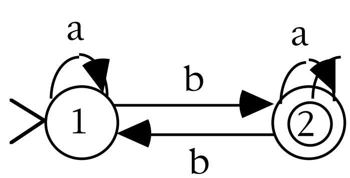
I.H. Assume each $R(i,j,n\text{-}1)$, $n \geq 1$ is regular.

Now consider $R(i,j,n)$

$$R(i,j,n) = R(i,j,n\text{-}1) \cup R(i,n,n\text{-}1)\ R(n,n,n\text{-}1)^*\ R(n,j,n\text{-}1)$$

Since each of these is regular, so is $R(i,j,n)$

Example.



$R(1,2,2) = R(1,2,1) \cup R(1,2,1)R(2,2,1)^*R(2,2,1)$

$R(1,2,1) = R(1,2,0) \cup R(1,1,0)R(1,1,0)^*R(1,2,0)$
$R(2,2,1) = R(2,2,0) \cup R(2,1,0)R(1,1,0)^*R(1,2,0)$

$R(1,2,0) = b$
$R(1,1,0) = a \cup e$
$R(2,2,0) = a \cup e$
$R(2,1,0) = b$

Substituting:

$R(1,2,1) = b \cup (a \cup e)(a \cup e)^*b = a^*b$
$R(2,2,1) = (a \cup e) \cup b(a \cup e)^*b = (a \cup e) \cup ba^*b$

$R(1,2,2) = a^*b \cup a^*b((a \cup e) \cup ba^*b)^*((a \cup e) \cup ba^*b) = a^*b(a \cup ba^*b)^*$